ON PARALLELISMS OF ODD DIMENSIONAL FINITE PROJECTIVE SPACES

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Abstract

In this paper we give a new, simple and explicit proof of a theorem of Baker. This states that every odd dimensional projective space over the field of two elements admits a 1-packing of 1-spreads, i.e. a partition of its lines into families of mutual disjoint lines whose union covers the space. This 1-packing may be generated from any one of its spreads by repeated application of a fixed collineation.

Introduction

A famous problem in recreational mathematics was Kirkman's fifteen schoolgirl problem in 1847: "Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily, so that no two will walk twice abreast." Since the introduction of this problem a great deal of mathematical activity has been devoted to finding resolvable designs. A pair $(X, \mathscr{B})$ is called a design or $t-(v, k, \lambda)$ *design* if

(i) X is a set with $|X| = v$. Elements of X are called *points*

(ii) B is a family of a subset of X, each $B \in \mathscr{B}$ satisfying $|B| = k$. Elements of $\mathscr{B}$ are called *blocks*

(iii) any subset of t point of X is contained in exactly λ blocks.

For example any complete graph on n points is a $2-(n, 2, 1)$ design.

A $t-(v, k, \lambda)$ design is called *t'-resolvable* if there exists a partition $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_m$ such that each $(X, \mathscr{B}i)$ is a $t'-(v, k, \lambda)$ design. Such a partition of $\mathscr{B}$ is called a *t'-resolution* of $(X, \mathscr{B})$.

The "15-schoolgirl-problem" was to find a 1-resolvable $2-(15, 3, 1)$ design, generally the "Kirkman problem" was to find all values of v such that 1-resolvable $2-(v, 3, 1)$ exists. This question was solved in 1971 by RAY-CHAUDURI and WILSON [5]. Otherwise the $2-(v, 3, 1)$ designs are called Steiner triple systems while a 1-resolvable $2-(v, 3, 1)$ design is called a Kirkman design.

Another very old problem asks for a decomposition of all k element subsets of a v-set (k dividing v) into partitions, i.e. show there exist 1-resolvable $k-(kn, k, 1)$ design. The case of k equal to two asks, in the language of graph theory, to show that the complete graph on an even number of vertices admits a 1-factorization. The general question was solved by BARANYAI [2].

Another type of these questions is the resolvability of finite projective spaces. We give some more notations and definitions before the details.

$GF(q)$ denotes the Galois field of q elements, where q is a prime power. $GF(q)^+$ denotes the non-zero elements of $GF(q)$. $V(n, q)$ denotes the n-dimensional vector space over $GF(q)$.

$AG(n, q)$ denotes the n-dimensional affine geometry over $GF(q)$. It has the vectors of $V(n, q)$ as points, and cosets of (d-dimensional) linear subspaces as its (d-dimensional) affine subspaces.

$PG(n, q)$ denotes the n-dimensional projective geometry over $GF(q)$. Its points are 1-dimensional subspaces of $V(n+1, q)$, lines are 2-dimensional subspaces, etc. Incidence is containment.

A *d-spread* of $PG(n, q)$ is a family of d-dimensional subspaces which are mutually disjoint and whose union is all of $PG(n, q)$. A *d-packing* of $PG(n, q)$ is a partition of all d-dimensional subspaces into d-spreads.

It is clear that $PG(n, q)$ is a $2-(q^n+q^{n+1}+\ldots+q+1,\ q+1, 1)$ design and if $PG(n, q)$ has a 1-packing then it is a 1-resolvable design. Simple numerical constraints show that the necessary condition for a packing to exist is that n be odd. Constructing packings in projective spaces appears to be a difficult problem and, to date, the only existence results known are:

(1) There exists a packing in $PG(2m+1, 2)$, m any positive integer (R. D. BAKER [1])

(2) There exists a packing in $PG(3, q)$ for all q a prime power (R. F. DENNISTON [4])

(3) There exists a packing in $PG(2^i-1, q)$ for all $i \geq 2$ and q a prime power (A. BEUTELSPACHER [3]). (This includes and generalizes [2]).

In this paper we give a new construction and proof of [1].

Construction

The points of $PG(2m+1, 2)$ are represented by the 1-dimensional subspaces of the space $V(2m+2, 2)$ which may be identified by the points of $V(2m+2, 2)\setminus\{0\}$. To use that $GF(2^{2m+2}) \cong GF(2^{2m+1}) \times GF(2)$ in our construction we represent the points of $PG(2m+1, 2)$ by pairs (x, y) where $x \in GF(2^{2m+1})$, $y \in GF(2)$ and $(x, y) \neq (0, 0)$. If $(x_1, y_1) \neq (x_2, y_2)$ then (x_1, y_1), (x_2, y_2) (x_1+x_2, y_1+y_2) are three points of a line. If $g \in GF(2^{2m+1})^+$ is a primitive element, then $(x, y) \to (gx, y)$ is a collineation of order $2^{2m+1}-1$. The three orbits of this collineation are:

$$\{(0, 1)\},$$

$$\{(x, 1)\colon\ x \in GF(2^{2m+1})^+\},$$

$$\{(x, 0)\colon\ x \in GF(2^{2m+1})^+\}.$$

The packing $\mathcal{P}$ will be cyclic in the sense that this collineation g changes the spreads cyclically, that is if $\mathcal{S}_1$ is a spread and $\mathcal{S}_{g^n}$ denotes the image of $\mathcal{S}_1$ by g^n then $\mathcal{P}$ contains the spreads $\mathcal{S}_{g^n}$, where $n = 0, 1, \ldots, 2^{2m+1} - 2$. Such a spread $\mathcal{S}_c(c \in GF(2^{2m+1})^+)$ contains the different lines of the next three types of lines.

a) $\{(0, 1), (c, 1), (c, 0)\}$,

b) $\left\{\left(\frac{c}{x}, 1\right), \left(\frac{c}{x+1}, 1\right), \left(\frac{c}{x^2+x}, 0\right)\right\}$ where $x \neq 0, 1$,

c) $\left\{\left(\frac{c}{x^2+x+1}, 0\right), \left(\frac{c}{y^2+y+1}, 0\right), \left(\frac{c}{z^2+z+1}, 0\right)\right\}$,

where $x, y, z \neq 0, 1$ and $y = \frac{1}{z+1}$, $z = \frac{1}{y+1}$ $\left(\text{and so } x = \frac{1}{z+1}\right)$.

Expressing the terms of c) by x we get $\{(c/(x^2 + x + 1), 0), (c(x^2 + 1)/(x^2 + x + 1), 0), (cx^2/(x^2 + x + 1), 0)\}$, where $x \neq 0, 1$.

Main result

Theorem. *If $\mathcal{S}_c$ contains the different lines listed in a), b), and c) then $\mathcal{S}_c$ is a spread, and the set of the spreads $\mathcal{S}_{g^n}$, where $n = 0, 1, \ldots, 2^{2m+1} - 2$ is a packing.*

Proof. First we prove that $\mathcal{S}_c$ is a spread that is if two lines of it have a common point then they are not different lines. If these lines are from the set of type b) then $c/x = c/(y + 1)$ implies that $c/(x + 1) = c/y$ and $c/(x^2 + x) = c/(y^2 + y)$, and similarly $c/(x^2 + x) = c/(y^2 + y)$ implies that $x = y$ or $x = y + 1$. If the lines are from c) then $c/(x^2 + x + 1) = c/(x'^2 + x' + 1)$ implies that $x = x'$. A line from a) and another from b) have no common point because $c = c/(x^2 + x)$ means that $x^2 + x + 1 = 0$ so $x^3 = 1$ that is x is a cube root of 1, which is impossible since 3 does not divide $2^{2m+1} - 1$. Because of this fact a line from b) and another from c) also have no common point since $c/(x^2 + x) = c/(y^2 + y + 1)$ implies that $(x + y)^2 + (x + y) + 1 = 0$. A line from a) and c) clearly have no common point. Next we prove that $\mathcal{S}_c$ and $\mathcal{S}_d$ have no common lines if $c \neq d$. Because of the last coordinate of the points it is enough to discuss the lines of the same type. Two lines of type a), namely $\{(0, 1), (c, 1), (c, 0)\}$ and $\{(0, 1), (d, 1), (d, 0)\}$ are clearly different. If $c/x = d/y$ then $c/(x + 1) \neq d/(y + 1)$ so two lines of type b) are also different. Similarly $c/(x^2 + x + 1) = d/(y^2 + y + 1)$ implies that $c(x^2 + 1)/(x^2 + x + 1) \neq d(y^2 + 1)/(y^2 + y + 1)$ so two lines of type c) are different. And this completes the proof.

Remark. The construction of BAKER [1] is implicit in the sense that we cannot get the lines of a spread without checking all the lines of the space whether the line is belonging to the spread or not. The construction of this paper is explicit, that is we can list the lines of a spread directly. Both of the constructions are cyclic but an easy calculation shows that they are different constructions.

Geometric demonstration of the construction

The main idea of the construction can be demonstrated in a simple geometric way. Let $g \in GF(2^{2m+1})^{+}$ be a primitive element and let us denote the point (x, y) by n_y if $x = g^n$, where $y = 0$ or 1, and denote the point $(0, 1)$ by ∞. We take two regular $(2^{2m+1} - 1)$-gons (R_0 and R_1) and index the vertices of R_y cyclically by n_y. Figure 1 presents this process when $m = 1$, and shows the pairs $\{a_y, b_y\}(y = 0$ or $1)$ for which $0_0, a_y, b_y$ are on a line of $PG(3, 2)$. $0_0 = (1, 0)$ so $b = a + 1$. The idea of the construction is to reflect R_1 about the vertical axis of symmetry (see R_1 on Figure 1 and Figure 2)

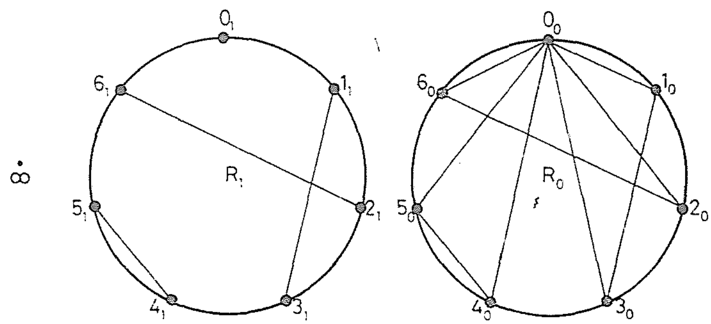

Fig. 1. Lines through 0_0 in $PG(3, 2)$

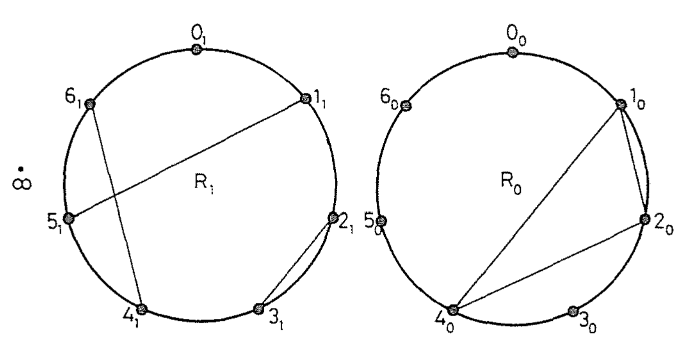

Fig. 2. A spread in $PG(3, 2)$

and observe that the lines of $PG(3, 2)$ through the reflected point-pairs are skew. The reflected images of a_1 and $(a+1)_1$ are $(1/a)_1$ and $(1/(a+1))_1$, respectively, so the third point of this line is $(1/(a^2+a))_0$. From these skew lines we get a spread (Figure 2). The lines of this spread are

$$\{\infty, 0_1, 0_0\}, \ \{1_1, 5_1, 6_0\}, \ \{2_1, 3_1, 5_0\}, \ \{4_1, 6_1, 3_0\}, \ \{1_0, 2_0, 4_0\}.$$

We get the other spreads of a packing by rotating R_0 and R_1 simultaneously. The reader can check this using the fact that if $\{a_0, b_0, c_0\}$ is a line then the length of the sides of the appropriate triangle in R_0 are 1, 2 and 3 (measured in arc length and if $\{a_i, b_j, c_k\}$ $(i, j, k = 0, 1)$ is a line then $\{a_0, b_0, c_0\}$ is also a line. Figure 3 and Figure 4 show the same process when $m = 2$.

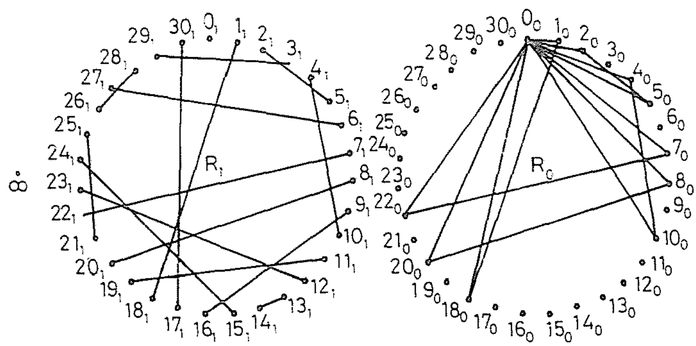

Fig. 3. Lines through 0_0 in $PG(5, 2)$

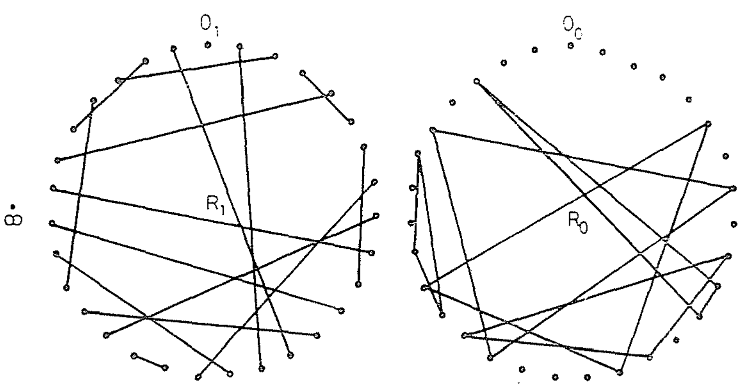

Fig. 4. A spread in $PG(5, 2)$

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