RANDOM BOUNDARY-INITIAL VALUE PROBLEMS FOR PARABOLIC DIFFERENTIAL EQUATIONS – ANALYTICAL AND SIMULATION RESULTS

B. FELLENBERG and J. vom SCHEIDT

Ingenieurhochschule Zwickau

Received June 20, 1988

Introduction

In recent years the application of mathematical methods in the natural and technological sciences is characterized among others by considering the stochastic nature of the interesting problem. This results in random equations with some random functions which determine the problem investigated.

In this paper we are concerned with boundary-initial value problems for parabolic differential equations with random inhomogeneous terms, initial or boundary conditions. Thereby, we consider especially these random influences in the form of weakly correlated functions describing real physical phenomena. These functions can be considered as functions without "distant effect" or as functions of "noise-natured character". That means, the influence of the random function does not reach far and its values at two points do not correlate when the distance of these points exceeds a certain quantity $\varepsilon > 0$. This number is called the correlation length and it is assumed to be sufficiently small in applications. Hence the correlation function R_ε fulfills the relation $R_\varepsilon(x, y) = 0$ if $|x - y| >= \varepsilon$. For an exact definition of weakly correlated functions see [5, 8]. An essential advantage of the use of these functions is the use of results of limit theorems for the random solution of the problem mentioned above.

In chapter 1 we consider a general random boundary-initial value problem and its almost sure (a.s.) solution. The theoretical results of the weakly correlated theory used in this paper are given in chapter 2. By application of them we analyse in chapter 3 some special problems in the space and half-plane to investigate the general effects of random inhomogeneous terms and conditions. Large investigations with respect to bounded domains with a random boundary condition can be found in [2, 3]. In chapter 4 we briefly repeat them and we give additionally comparison results obtained by physical and mathematical simulation.

1. General problem and its a.s. solution

We consider the following boundary-initial value problem of heat conduction (w denotes the random input functions)

$$u_t = a \Delta u + f_Q(x, t, w), \quad (x, t) \in Gx(0, \infty)$$

$$u(x, 0) = f_A(x, w), \quad x \in \bar{G} = G \cup \partial G$$

$$A_k u_{n_k}(x, t) + B_k u(x, t) = f_{R,k}(x, t, w), \ (x, t) \in \partial G_k x(0, \infty), \qquad k = 1, 2, \ldots, n,$$

where $G \subset R^m (m \leqq 3)$ is a simply connected domain with piecewise smooth boundary $\partial G = \bigcup_{k=1}^{n} \partial G_k$ and n_k denotes the outward normal to G_k. Further, the compatibility and the continuity conditions of the above problem are assumed to be fulfilled.

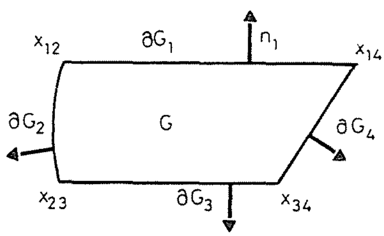

Fig. 1. Domain G

Then, the a.s. solution of (1), if it exists, possesses the form with the Greens function $G(x, t; x_0, t_0)$

$$u(x, t, w) = \int_0^t \int_G G(x, t; x_0, t_0) f_Q(x_0, t_0, w)\, dx_0 dt_0 + \int_G G(x, t; x_0, 0) f_A(x_0, w)\, dx_0 - \tag{2}$$

$$- a \int_0^t \sum_{k \in K_A} \int_{\partial G_k} \frac{f_{R,k}(x_0, t_0, w)}{B_k} \frac{\partial G(x, t; x_0, t_0)}{\partial n_k(x_0)} dS(x_0)\, dt_0 +$$

$$+ a \int_0^t \sum_{k \in K_B} \int_{\partial G_k} \frac{f_{R,k}(x_0, t_0, w)}{A_k} G(x, t; x_0, t_0)\, dS(x_0)\, dt_0$$

with $K_A = \{k \in \{1, \ldots, n\} : A_k = 0\}$, $K_B = \{1, \ldots, n\} - K_A$.

This statement is obtained by generalization of results in [1, 9].

Examples:

1. $G = R^m$; then

$$G(x, t; x_0, t_0) = \frac{H(t-t_0)}{(4\pi a(t - t_0))^{m/2}} \exp\left(- \frac{|x - x_0|^2}{4a(t - t_0)}\right) \tag{3}$$

with the Heaviside-function H.

2. $G = R^+ = \{x > 0\}$; here

$$G(x, t; x_0, t_0) = \frac{H(t - t_0)}{(4\pi a(t - t_0))^{1/2}} \left(\exp\left(- \frac{(x - x_0)^2}{4a(t - t_0)}\right) + \exp\left(- \frac{(x + x_0)^2}{4a(t - t_0)}\right)\right) \tag{4}$$

for the boundary condition $\frac{\partial}{\partial x} G(0, t; x_0, t_0) = 0$, i. e. $B = 0$.

3. For bounded domains GR^m we obtain the series statement

$$G(x, t; x_0, t_0) = \sum_{i=1}^{\infty} f_i(x) f_i(x_0) \exp\left(- a\Lambda_i(t - t_0)\right) \tag{5}$$

where Λ_i and f_i are the eigenvalues and eigenfunctions of the adjoining eigenvalue problem with respect to the operator $-\Delta$. For the stochastic analysis we can separate the problem (1) by

$$\begin{aligned} f_Q(x, t, w) &= \langle f_Q(x, t)\rangle + \bar{f}_Q(x, t, w) \\ f_A(x, w) &= \langle f_A(x)\rangle + \bar{f}_A(x, w) \\ f_{R,k}(x, t, w) &= \langle f_{R,k}(x, t)\rangle + \bar{f}_{R,k}(x, t, w), \quad k = 1, \ldots, n, \end{aligned} \tag{6}$$

and

$$u(x, t, w) = \langle u(x, t)\rangle + \bar{u}(x, t, w).$$

Then, we obtain for $\bar{u}$ the presentation (2) with $\bar{f}_.$ instead of $f_.$.

2. Theoretical results of the weakly correlated theory

Random linear functionals of the form

$$r_{ij\varepsilon}(w) = \int_{D_i} F_{ij}(x) f_{j\varepsilon}(x, w)\, dx, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, l$$

as well as

$$r_{i\varepsilon}(w) = \sum_{j=1}^{1} r_{ij\varepsilon}(w), \quad i = 1, 2, \ldots, n,$$

are considered where $(f_{1\varepsilon}(x, w), \dots, f_{1\varepsilon}(x, w))^T$ is an a.s. continuous, weakly correlated connected vector function with $D \subset R^m$. The domains D_i of $D = \bigcup_{i=1}^{n} D_i$ possesses a piecewise smooth boundary.

The essential object of the application of the theory is the following limit theorems:

— Let $F_{ij}(x) \in L_1(D) \cap L_2(D)$, where $F_{ij}(x) = 0$ for $x \in D - D_i$, and $\langle f_{j\varepsilon}^2(x)\rangle \leqq c_2 < \infty, j = 1, \dots, l$, then it yields

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^m} \langle r_{i_1 j_1} r_{i_2 j_2 \varepsilon}\rangle = {}^2A_1(F_{i_1 j_1}, F_{i_2 j_2}) \tag{7}$$

with

$${}^2A_1(F_{i_1 j_1}, F_{i_2 j_2}) = \int_{D_{i_1} \cap D_{i_2}} F_{i_1 j_1}(x) F_{i_2 j_2}(x)\, a_{i_1 j_2}(x)\, dx$$

Thereby the intensity $a_{ij}(x)$ between $f_{i\varepsilon}$ and $f_{j\varepsilon}$ is given by

$$a_{ij}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^m} \int_{K_\varepsilon(0)} \langle f_{i\varepsilon}(x)\, f_{j\varepsilon}(x + z)\rangle\, dz$$

with $K_\varepsilon(x) = \{z \in R^m = |x - z| \leqq \varepsilon\}$.

— If further $\langle |f_{j\varepsilon}(x)|P\rangle \leqq c_p < \infty$, $j = 1, \dots, 1$ and $p = 1, 2, \dots,$ then it yields in distribution

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{m/2}} (r_{1\varepsilon}(w), \dots, r_{n\varepsilon}(w)^T = (g_1(w), \dots, g_n(w)^T = g(w) \tag{8}$$

where $g(w)$ is normally distributed with $\langle g\rangle = 0$ and correlation relations

$$\langle g_p g_q\rangle = \sum_{j_1, j_2 = 1}^{1} {}^2A_1(F_{pj_1}, F_{qj_2}),\ p, q = 1, \dots, n.$$

That means practically for small ε $r_{i\varepsilon}(w)$ is approximately normally distributed and for the covariances it yields

$$\langle r_{pj_1\varepsilon} r_{qj_2\varepsilon}\rangle \approx {}^2A_1(F_{pj_1}, F_{qj_2})\, \varepsilon^m.$$

In the recent years these results were improved by determining the deviation of the normal distribution and the next terms in the expansion of the moments (see e.g. [5, 7]). We restrict us here to the result for the variance. For this $|F_{ij}(x)| \leqq c < \infty$, for all $x \in D_i$, $i = 1, \dots, n$ and $j = 1, \dots, l$, is assumed and the correlation function possesses an expansion in the following sense

$$\int_{K\varepsilon(0)} \langle f_{i\varepsilon}(x) f_{j\varepsilon}(x + z)\rangle dz = a_{ij}(x)\, \varepsilon^m + b_{ij}(x)\, \varepsilon^{m+1} + o(\varepsilon^{m+1})$$

where $a_{ij}(x)$ is continuous, $b_{ij}(x)$ is bounded and $o(\varepsilon^{m+1})$ is uniform with respect to x.

Then it yields

$$\langle r_{ij\varepsilon}^2 \rangle = {}^2A_1(F_{ij}, F_{ij})\,\varepsilon^m + {}^2A_2(F_{ij}, F_{ij})\,\varepsilon^{m+1} + o(\varepsilon^{m+1}) \tag{9}$$

where

$${}^2A_2(F_{ij}, F_{ij}) = \int_{D_i} F_{ij}^2(x) b_{jj}(x)\, dx + \int_{D_i} F_{ij}^2(x)\, S(x; 0, 1)\, dS(x),$$

and $S(x; 0, 1)$ is a further characteristic of $f_{j\varepsilon}(x, w)$ concerned with the behaviour at the boundary of D_i. Often the relation $b_{jj}(x) = 0$ is true for the considered correlation function of $f_{j\varepsilon}$.

3. Problems in the space and in the half plane

At first, we consider (1) in the form

$$\begin{aligned} u_t &= a\Delta u + f_Q(x, t, w), \ (x, t) \in R^m x(0, \infty) \\ u(x, 0) &= f_A(x, w), \qquad x \in R^m. \end{aligned} \tag{10}$$

If we have $\bar{f}_Q(x, t, w) = 0$, and $\bar{f}_A(x, w)$ is weakly correlated with the according properties mentioned above we obtain with (2), (3), (6) and (7), (8)

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{m/2}} \bar{u}(x, t, w) = g(x, t, w) \qquad \text{in distribution,}$$

where g is a centred Gaussian field with

$$\langle g(x_1, t_1)\, g(x_2, t_2) \rangle = \frac{1}{(4\pi a)^m (t_1 t_2)^{m/2}} \cdot$$

$$\cdot \int_{R^m} \exp\left(-\frac{1}{4a}\left(\frac{|x_1 - x_0|^2}{t_1} + \frac{|x_2 - x_0|^2}{t_2}\right)\right) a_A(x_0)\, dx_0$$

and $a_A(x)$ denotes the intensity of $\bar{f}_A$. For a homogeneous initial temperature field it is $a_A(x_0) = a_A = \text{const}$, and especially the variance is given by

$$\langle g^2(x, t) \rangle = \frac{a_A}{(4\pi a t)^m} \int_{R^m} \exp\left(-\frac{1}{2at}|x - x_0|^2\right) dx_0 = \frac{a_A}{\sqrt{8\pi a t^m}} = {}^2A_1$$

Moreover, ${}^2A_2 = 0$ holds for $b_A (= b_{jj}) = 0$. Hence we have in this case

$$\langle \bar{u}^2(x, t) \rangle \approx a_A \varepsilon^m | (8\pi a t)^{m/2} = v(t)$$

up to the order $o(\varepsilon^{m+1})$.

Figure 2 shows for $m = 1$ the behaviour of the approximated normalized dispersion in comparison with the exact results. Thereby $a_A = \sigma^2 = \langle \bar{f}_A^2(x) \rangle$ is used and it has to be noted that the exact results could only be determined by numerical integration methods. We can state a good agreement after a short, with ε decreasing, time t.

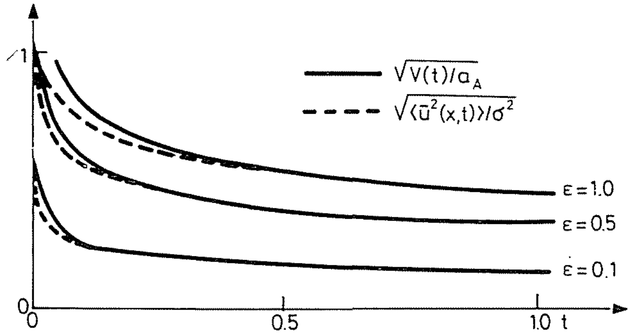

Fig. 2. Exact and approximate standard deviation

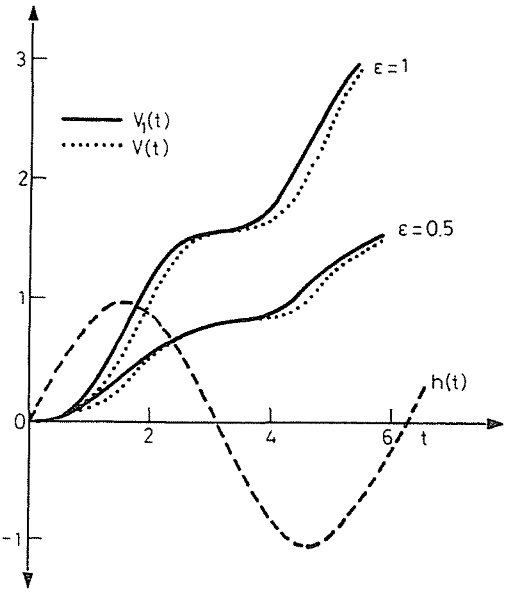

Fig. 3. 1st and 2nd order approximation of the variance

Figure 3 gives an impression for the behaviour of the variance in the case of random sources $\bar{f}_Q$ but deterministic initial temperatures, i.e. $\bar{f}_A=0$. Thereby $\bar{f}_Q(x,t,w)=\sin(t)\bar{f}_1(t,w)$ was considered where $\bar{f}_1(t,w)$ was assumed to be stationary with intensity $a_1=\sigma^2=\langle\bar{f}_1^2(t)\rangle=1$ and $S(x;0,\ 1)=-\dfrac{\sigma^2}{6}$. Moreover, the abbreviations V, V_1 were used in the following sense

$$\langle\bar{u}^2(x,t)\rangle \approx \sigma^2\left(\frac{t}{2}-\frac{1}{4}\sin(2t)\right)\varepsilon-\frac{\sigma^2}{6}\sin^2(t)\varepsilon^2=$$
$$=[V_1(t)+V_2(t)]V(t)$$

Remark: The coupling of the two cases considered above is possible, if e.g. $\bar{f}_Q(x, t, w) = h(t)\bar{f}_2(x, w)$ and $(\bar{f}_2(x, w), \bar{f}_A(x, w))^T$ is weakly correlated connected.

Secondly, we consider the following problem in the half plane:

$$u_t = a\Delta u$$

$$u(x, y, o) = u_0(x, y)$$

$$u_y(x, o, t) = f_R(t, w)$$

in the domain $G = R \times R^+$.

Then we have for $\bar{u} = \bar{u}(y, t, w)$ the simpler problem

$$\bar{u}_t = a\Delta u_{yy}, \quad \bar{u}(y, 0) = 0$$

$$\bar{u}_y(0, t) = \bar{f}_R(t, w) = f_R(t, w) - \langle f_R(t) \rangle$$

having the solution (cf. (4))

$$\bar{u}(y, t, w) = -\sqrt{\frac{a}{\pi}} \int_0^t \frac{\exp\left(-\frac{y^2}{4a(t-t_0)}\right)}{\sqrt{t-t_0}} \bar{f}_R(t_0, w)dt_0 =$$

$$= \int_0^t F(t - t_0, y)\,\bar{f}_R(t_0, w)\,dt_0$$

where $F \in L_{2,t_0}(0, t)$ for all $y > 0$.

By using the limit statement (7) we approximately obtain for a weakly correlated $\bar{f}_R(t, w)$ the variance

$$\langle \bar{u}^2(y, t) \rangle \approx -\frac{aa_1\varepsilon}{\pi} Ei\left(-\frac{y^2}{2at}\right)$$

where a_1 denotes the constant intensity of $\bar{f}_R$ and Ei(.) is the exponential integral function. Considering the compatibility condition $\bar{f}_R(0, w) = 0$ a.s., the intensity cannot be constant. Figure 4 shows the deviation from the instationary case where the smoothing function

$$h(t) = \begin{cases} t/c & \text{for} \quad 0 \leq t \leq c \\ 1 & \text{for} \quad t > c \end{cases}$$

is used for $\bar{f}_R(t, w) = h(t)\, f_1(t, w)$ and f_1 possesses the intensity a_1.

We can state that with increasing times t or decreasing values of c the compatibility condition can be neglected.

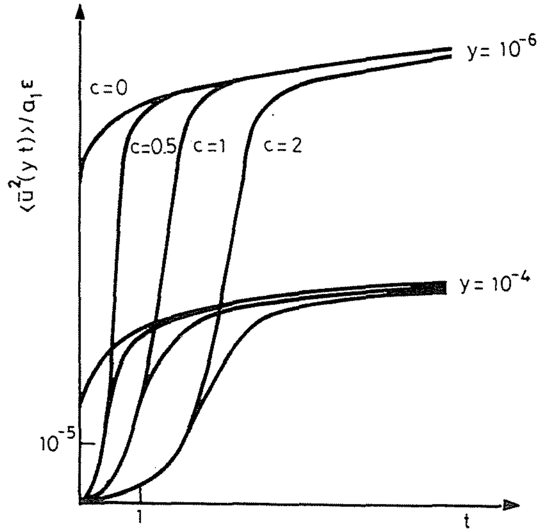

Fig. 4. Normalized variance in dependence on c

4. Analytical and simulation results for a bounded domain

Now we turn to a problem in a bounded domain G

$$G = \{x, y): |x| < R, 0 < y < L\}$$

with an inhomogeneous random boundary condition

$$\bar{u}_t = a\Delta u, \ \bar{u}(x, y, 0) = 0$$

$$\bar{u}_y(x, y, t)\big|_{(\partial G)_1} = \bar{f}_R(x, t, w) = f_R(x, t, w) - \langle f_R(x, t)\rangle$$

$$[\bar{u}_{n_k}(x, y, t) + \alpha_k \bar{u}(x, y, t)]_{(\partial G)_k} = 0, \ k = 2, 3, 4$$

where G_k, $k = 1, \ldots, 4$, denotes the boundaries for $y = 0$, $x = -R$, $y = L$ and $x = R$.

By using (2), (5) and (6) and "averaged" solution $\bar{u}_h(x, y, t, w) = (\bar{u}, w_h)_G$ with an averaging kernel $w_h \in C_0^\infty(G)$, $|w_h| = 1$ and h sufficiently small can be obtained in the form (cf. (5))

$$\bar{u}_h(x, y, t, w) = \int_0^t \int_{-R}^{R} F(x_0, t - t_0)\bar{f}_R(x_0, t_0, w)\, dx_0 dt_0$$

with

$$F(x_0, t - t_0) = -a \sum_{k,l}^{\infty} f_{2l}(0) f_{1k}(x_0) \exp\left(-a\Lambda_{kl}(t - t_0)(f_{kl}, w_h)\right)$$

possessing strong convergence properties (see [2, 3]).

Assuming again a constant intensity we obtain for a weakly correlated function $\bar{f}_R(x, t, w)$ with respect to (x, t)

$$\langle \bar{u}(x_1, y_1, tj)\, \bar{u}(x_2, y_2, t_2)\rangle = {}^2A_1\varepsilon^2 + A_2\varepsilon^3 + o(\varepsilon^3)$$

for small h where

$${}^2A_1 = aa_R \sum_{k,l,m=0}^{\infty} f_{kl}(0)\, f_{km}(0)\, \frac{(f_{kl}, w_1)(f_{km}, w_2)}{\Lambda_{kl} + \Lambda_{km}}$$

$$\big(\exp(-\, a(\Lambda_{kl}(t_1 - t_{12}) + \Lambda_{km}(t_2 - t_{12}))) - \exp\,(-\, a\Lambda_{kl}t_1 + \Lambda_{km}t_2)\big) +$$
$$+ \big(at_{12}/(2L^2R^2)\big)\delta_{\alpha_2\alpha_3\alpha_4 0}$$

$$\text{with } \delta_{\alpha_2\alpha_3\alpha_4 0} = \begin{cases} 1 \text{ for } \alpha_2 = \alpha_3 = \alpha_4 = 0 \\ 0 \text{ otherwise.} \end{cases}$$

And, for $b_R = 0$ it can be shown that ${}^2A_2 = 0$, too.

This approximation result could be confirmed by physical measurements with electroanalogical model-simulation for some realistic values. A good agreement between the measured (x) values and ours (- - -) can be seen in Fig. 5 where two correlation functions $R_{2\varepsilon}$ are used. Especially it was $\varepsilon/2R =$ $= 0.07$ and $\varepsilon/t = 0.15$.

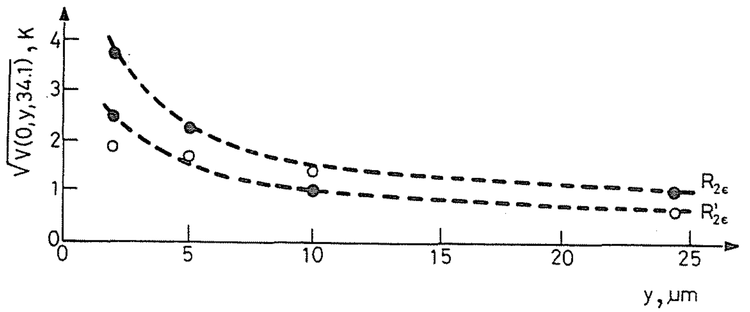

Fig. 5. Measured and approximated standard deviations

If $\bar{f}_R(x, t, w) = cf_1(t, w)$ where the compatibility conditions are neglected we obtain

$$\langle \bar{u}(x_1, y_1, t_1)\bar{u}(x_2, y_2, t_2)\rangle \approx aa_1 \sum_{k,h,l,m}^{\infty} f_{21}(0) f_{2m}(0)\, W_k W_h$$

$$\frac{(f_{kl}, w_1)(f_{hm}, w_2)}{\Lambda_{kl} + \Lambda_{hm}} \big(\exp -\, a\Lambda_{kl}(t_1 - t_{12}) + \Lambda_{hm}(t_2 - t_{12}))\big) -$$
$$-\exp\big(-\,(a\Lambda_{kl}t_1 + \Lambda_{hm}t_2)\big)$$

where $W_k = c \int_{-R}^{R} f_{1k}(x)\, dx$.

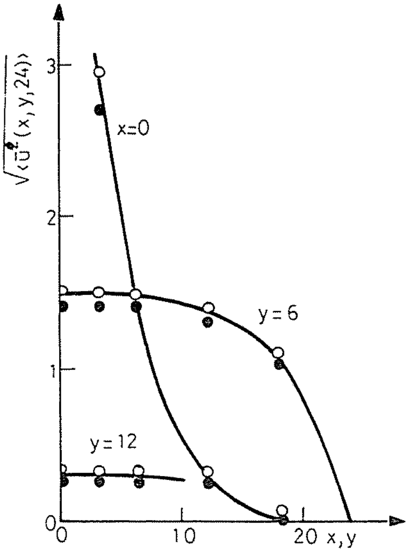

Fig. 6. Simulated and approximated standard deviations

This result was confirmed by a mathematical simulation of the weakly correlated function $f_1(t, w)$. Thereby the simulation procedure described in [6] was used. Figure 6 shows the simulated standard deviations (x and o) of two simulation series in comparison with our approximated (—) ones. Here, it was $c = 1$ and $\varepsilon/t = 0.21$.

References

1. BECUS, G. A.: Random generalized Solutions to the Heat Equation. J. Math. Anal. Appl. *60* (1977), 93—102.
2. FELLENBERG, B. and VOM SCHEIDT, J.: Heat Propagation with a Random Sliding Input of Heat. Bull. Appl. Math. *XLVI* (1987), 65—78.
3. FELLENBERG, B. and VOM SCHEIDT, J.: On a Random Boundary Initial Value Problem for Parabolic Differential Equations. Bull. Appl. Math. *XLV* (1987), 103—122.
4. JUNG, M.: Das Programmsystem FEMGP zur Loesung elliptischer und parabolischer Differentialgleichungen. Programmdokumentation, TUK 1987.
5. VOM SCHEIDT, J.: Random Equations of the Matematical Physics. Akademie-Verlag Berlin, im Druck.
6. VOM SCHEIDT, J. and FELLENBERG, B.: Some simulation results as to weakly correlated processes. Math. Comp. Simul. *29* (1987), 191—208.
7. VOM SCHEIDT, J. and FELLENBERG, B.: On the Distribution of Functionals of Stochastic Fields, in: Stochastic Differential Systems (Eds. H. J. ENGELBERT and W. SCHMIDT), Proc. of IFIP-Working Conference, Springer-Verlag 1986.
8. VOM SCHEIDT, J. and PURKERT, W.: Random Eigenvalue Problems. Akademie-Verlag Berlin und North-Holland New York, 1983.
9. STAKGOLD, I.: Boundary Value Problems of Mathematical Physics, Vol. *II*, Macmillan, New York, 1967.

B. FELLENBERG
J. VOM SCHEIDT

Ingenieurhochschule Zwickau
PSF 35
Zwickau, BRD 9541