PROBLEMS OF INCLUSION AND EQUIVALENCE OF DIGITAL SYSTEMS

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I. Introduction

The theory of boolean spaces became a useful engineering tool for the design of digital systems, both combinational and sequential ones. It was also shown that well-known methods of boolean functions minimization like Karnaugh's maps and Quine's method can be easily drawn following the principles of boolean spaces [2], [3]. This work has been done in order to make this tool even more flexible especially for designing very bulky systems. The question can be stated as follows. Suppose there is the matrix model of the digital system to be designed but in advance we have some additional information concerning the properties of an input matrix or acceptable forms of boolean functions. Can we use this information for diminishing the dimensions of the model? In the paper it is shown how to see the problem more formally and how we can find its solution in some special cases. The idea to diminish the model of a digital system to be designed is not new. Some results of partitional automata [1] allow to represent the defined model by a less bulky one if some additional properties of the model are known. However, the results of automata theory are mostly very far from the possibilities of direct practical implementation since the automata model seems to be too general for hardware design. The paper presented has a practical engineering application for diminishing the dimensions of the system model to be designed.

II. Preliminary

This is a very comprehensive description of the basic facts concerning the theory of boolean spaces [3].

Let x be an n dimensional boolean vector; it means that its coordinates take the value 1 or 0. For vectors of the same dimension n we introduce their summation and multiplication, as normal boolean operations, performed in componentwise manner. The vector negation, denoted by $\bar{\mathrm{x}}$, is a vector obtained from x by negation of its coordinates. Let X^n denote a set of all n dimensional boolean vectors. We distinguish two of them; the maximum vector denoted by 1 and minimum vector denoted by 0, are vectors all coordinates of which are equal to 1 or to 0, respectively. The system $\langle X^n, +, \cdot, -, \{1, 0\}\rangle$ is called n dimensional combinational boolean space, where $+, \cdot, -$ are the boolean operations performed on vectors. Any combinational boolean space is a boolean algebra. Let A be a subset of the set X^n; we present the set A in the form of a matrix whose every column corresponds to the boolean vector; the matrix **A** is called a boolean matrix. Suppose we have boolean matrix **A** and any n dimensional boolean vector **y**. We say that vector **y** is combinationally dependent on matrix **A** if there exists a function $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \mathbf{y}$, where vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ constitute the matrix **A** and the function is a composition of the boolean vector operations $+, \cdot, -$. We say that the matrix **B** is combinationally dependent on matrix **A** if and only if every vector $\mathbf{y} \in \mathbf{B}$ is combinationally dependent on **A**.

For a given matrix **A** and vector **y** we say that the j-th row of matrix **A** is of type one if and only if $y(j) = 1$, we denote this fact by $R_A(j) = T_1$. If $y(j) = 0$ then correspondingly the j-th row of **A** is called the row of type 0 and we denote it by $R_A(j) = T_0$.

Theorem 1

Vector **y** is combinationally dependent on matrix **A** if and only if its each row of type 1 is different from every row of type zero in this matrix, with respect to the vector **y** [2].

By submatrix **C** of matrix **A** we understand a matrix made of a subset of vectors of matrix **A**.

Theorem 2

For $y(j) = 1$ we can create an implicant taking the j-th row of submatrix **C** of the matrix **A** if and only if $R_C(j) \ne R_C(k)$, for all the k such that $y(k)=0$.

So the procedure of minimization of combinational dependence of boolean vector **y** on matrix **A** looks as follows.

Algorithm

1. For each $y(j) = 1$ find a submatrix **C** of matrix **A** such that $R_C(j) \ne R_C(k)$, if $y(k) = 0$.

2. Create an implicant P_j for the j-th row of subtable C satisfying the above condition.

3. For the set of the found implicants find the minimum cover. The above algorithm was first presented by KAZAKOW [4]. For two boolean vectors $\mathbf{x}_1, \mathbf{x}_2$ of the same dimension we introduce the operator of sequencing as follows:

$$\mathbf{x}_1 \bigcirc \mathbf{x}_2 = \mathbf{y} \text{ such that } \begin{cases} y(j) = 1 \text{ if } [x_1(j) = 1 \text{ or } y(j-1) = 1] \text{ and } \\ \qquad x_2(j) = 0 \\ y(j) = 0 \text{ otherwise} \end{cases}$$

The sequential boolean space is the following system $\langle X^n, +, \cdot, -, \{1, 0\}, \bigcirc \rangle$, so it is a combinational boolean space with the operator of sequencing added.

We say that boolean vector $\mathbf{y}$ is strongly sequentially dependent on matrix $\mathbf{A}$ if there exist two vectors $\mathbf{y}_1, \mathbf{y}_2$ such that $\mathbf{y} = \mathbf{y}_1 \bigcirc \mathbf{y}_2$ and if they both are combinationally dependent on $\mathbf{A}$.

For the given matrix $\mathbf{A}$ and vector $\mathbf{y}$ we say that the j-th row of $\mathbf{A}$ is of type B_1 if and only if $y(j) = 1$ and $y(j-1) = 0$ or $j = 1$ and $y(j) = 1$, we denote it by $R_A(j) = B_1$.

Similarly we say that the j-th row of matrix $\mathbf{A}$ is of type B_0 if and only if $y(j) = 0$ and $y(j-1) = 1$.

Theorem 3

Vector $\mathbf{y}$ is strongly sequentially dependent on matrix $\mathbf{A}$ if and only if each row of type B_1 is different from every row of type 0 and each row of type B_0 is different from every row of type one in matrix $\mathbf{A}$, with respect to vector $\mathbf{y}$ [2].

III. Partitional Dependence

Let π_X be a partition of matrix $\mathbf{X}$ into a set of submatrices $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_k$ such that for each $\mathbf{X}_i \neq \mathbf{X}_j$; $\mathbf{X}_i \cap \mathbf{X}_j = \emptyset$ and $\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_k = \mathbf{X}$.

It means that no vector belongs to any two submatrices but each vector belongs to any one submatrix.

Suppose now that there exists the partition π_X dividing matrix $\mathbf{X}$ into k submatrices. Moreover, matrix $\mathbf{B}$ is combinationally dependent on each submatrix $\mathbf{X}_i$.

For a given vector $\mathbf{y} \in \mathbf{B}$ we denote the disjunctive normal form of combinational dependence of vector $\mathbf{y}$ on submatrix $\mathbf{X}_i$ by $f^d(X_i)$. Accordingly by $f^c(X_i)$ we denote the conjunctive normal form of the combinational dependence of vector $\mathbf{y}$ on submatrix $\mathbf{X}_i$.

Definition 1

The disjunctive partitional dependence of a given vector y on a given matrix **X** at the partition π_X we call the following expression

$$\mathrm{y} = f^d(X_1) + f^d(X_2) + \ldots + f^d(X_k) = f^d_\pi(X).$$

Similarly by $\mathrm{y} = f^c(X_1) + f^c(X_2) + \ldots + f^c(X_k) = f^c_\pi(X)$ we denote the conjunctive partitional dependence of the vector y on matrix **X** at the partition π_X.

Definition 2

We mean by sequential disjunctive dependence of the given vector y on the given matrix **X** at the partition π_X the following expression

$$\mathrm{y} = [f^d_1(X_1) + f^d_1(X_2) + \ldots + f^d_1(X_k)] \bigcirc [f^d_2(X_1) + f^d_2(X_2) + \ldots + f^d_2(X_k)] = f^{Sd}_\pi(X)$$

where subexpressions in brackets represent correspondingly disjunctive partitional dependences of vectors y_1 and y_2 such that $\mathrm{y} = \mathrm{y}_1 \bigcirc \mathrm{y}_2$.

Accordingly we denote the sequential conjunctive dependence of a given vector y on given matrix **X** at the partition π_X by the following expression

$$\mathrm{y} = [r^c_1(X_1) + f^c_1(X_2) + \ldots + f^c_1(X_k)] \bigcirc [f^c_2(X_1) + f^c_2(X_2) + \ldots + f^c_2(X_k)] = f^{Sc}_\pi(X).$$

IV. Inclusion of Combinational Systems

Let Ω be a set of expressions presenting the dependence of matrix **Y** on matrix **X**.

By Γ we denote an arbitrary subset of Ω, it can be, for instance, a set of all disjunctive normal forms of the combinational dependence or the set of all conjunctive normal forms of it and so on.

Let $\Gamma^d_{\pi_X}$ mean the set of all the partitional disjunctive normal expressions presenting the combinational dependence of matrix **Y** on matrix **X**; correspondingly, $\Gamma^c_{\pi_X}$ means a set of all the partitional conjunctive normal forms of combinational dependence of these matrices.

Definition 3

By switching system we mean the following triple $\langle \mathbf{X}, \mathbf{Y}, \Gamma \rangle$, where **X** and **Y** mean the input and output matrix, respectively, and Γ represents a defined set of combinational dependences of **Y** on **X**.

If for a given system $\Gamma \equiv \Omega$, we omit it at the switching system specification and such a system is denoted by the pair $\langle \mathbf{X}, \mathbf{Y} \rangle$. Let the switching systems $\langle \mathbf{X}, \mathbf{Y}, \Gamma \rangle$ and $\langle V, Z, \Psi \rangle$ be given.

The following definition is valid for combinational systems and sequential ones as well. But if we speak about sequential systems then Γ and Ψ mean some expressions representing the sequential dependence of matrix **Y** on matrix **X** (for details see chapter V).

Definition 4

System $\langle \mathbf{X}, \mathbf{Y}, \Gamma \rangle$ is included in system $\langle V, Z, \Psi \rangle$ if there exist two one-to-one mappings h_1: $\mathbf{X} \to V$, h_2: $Y \to Z$ such that if $[Y = f(\mathbf{X})]$ and $f(\mathbf{X}) \in \Gamma$ then $Z = h_2(Y) = f(h_1[\mathbf{X}])$ and $f[h_1(\mathbf{X})] \in \Psi$.

Definition 5

We say that systems $\langle \mathbf{X}, \mathbf{Y}, \Gamma \rangle$ and $\langle V, Z, \Psi \rangle$ are equivalent if system $\langle \mathbf{X}, \mathbf{Y}, \Gamma \rangle$ is included in system $\langle V, Z, \Psi \rangle$ and system $\langle V, Z, \Psi \rangle$ is included in system $\langle \mathbf{X}, \mathbf{Y}, \Gamma \rangle$.

Let π and τ be two partitions on the set of indices $\{1, 2, \ldots, m\}$ such that every block of partition π is included in any block of partition τ.

To denote the one-to-one mapping h_1 we will use the following equivalent description: h_1: $\mathbf{X} \to V$; $V = h_1(\mathbf{X})$; $\mathbf{v}_j = h_1(\mathbf{x}_i)$. We denote by $\mathbf{X}_\pi$ a submatrix of matrix **X** such that the indices of its vectors belong to a block of partition π. Accordingly by V_τ is denoted a submatrix of matrix **V** such that the indices of its vectors belong to a block of partition τ.

Theorem 4

System $\langle \mathbf{X}, \mathbf{y}, \Gamma^d_{\pi_X} \rangle$ is included in system $\langle V, \mathbf{z}, \Gamma^d_{\tau_V} \rangle$ if and only if the following conditions are satisfied:

1. There is one-to-one mapping h_1: $\mathbf{X} \to \mathbf{V}$.

2. For each $y(j) = 1$ there exists a minimum matrix $\mathbf{X}_\pi$ such that $R_{X_\pi}(j) \nparallel R_{X_\pi}(k)$ whenever $y(k) = 0$.

3. There exists submatrix $\mathbf{V}_\tau$ and exists $z(l) = 1$ such that $\mathbf{V}_\tau = h_1(\mathbf{X}_\pi)$ and $R_{X_\pi}(j) = R_{V_\tau}(1)$.

4. $R_{V_\tau}(1) \nparallel R_{V_\tau}(u)$ whenever $z(u) = 0$.

Proof

Assumption 2 implies that the set $\Gamma^d_{\pi_X}$ is not empty.

Assumptions 1, 3 and 4 imply that if there exists a minimum disjunctive normal form of combinational dependence $f(\mathbf{X})$ such that $\mathbf{y} = f(\mathbf{X})$ then must exist a form of combinational dependence $z = f[h_1(\mathbf{X})]$.

If matrix $\mathbf{V}_\tau$ satisfying assumptions 3 and 4 is included in any matrix $\mathbf{V}_\tau^1$, then $\mathbf{V}_\tau^1$ also satisfies these assumptions. Therefore for any $f(\mathbf{X}) = \mathbf{y}$ disjunctive normal form of combinational dependence of vector y on matrix **X** there exists a disjunctive normal form $f[h_1(\mathbf{X})] = \mathbf{z}$.

Therefore $\Gamma_{\pi_X}^d$ is included in $\Gamma_{\tau_V}^d$ so the system $\langle \mathbf{X}, \mathbf{y}, \Gamma_{\pi_X}^d \rangle$ is included in system $\langle \mathbf{V}, \mathbf{z}, \Gamma_{\tau_V}^d \rangle$.

We can extend the theorem 4 as follows:

Let $\Gamma_{\Pi_X}^q$ be a set of all the normal disjunctive forms of combinational dependence such that any implicant in there has no less than q laterals.

Substituting $\Gamma_{\pi_X}^q$ for $\Gamma_{\pi_X}^d$ we can modify theorem 4 by replacing assumption 2 by the following one.

2. For each $y(j) = 1$ there exists a submatrix $\mathbf{X}_\pi$ with no less than q columns such that $R_{X_\pi}(j) \mp R_{X_\pi}(k)$ whenever $y(k) = 0$.

The proof of the so modified theorem 4 can be done by analogy to the proof of the original theorem. This modified theorem can be used in order to diminish the model of a combinational system which is to be implemented with demultiplexers and decoders.

The other analogy we can draw by replacing a set $\Gamma_{\pi_X}^d$ by the set of all the normal conjunctive forms $\Gamma_{\pi_X}^c$.

If we want to extend theorem 4 on multi-output systems then we have to add the following assumptions

(i) There exists a one-to-one mapping h_2 such that $Z = h_2(\mathbf{Y})$.

(ii) For each vector $\mathbf{y} \in \mathbf{Y}$ there exists vector $\mathbf{z} = h_2(y)$ satisfying the assumption of theorem 4.

V. Inclusion of Sequential Systems

Similarly as it was done for combinational systems we introduce the classes of sequential dependences.

Let Ω^S be now a set of all the expressions representing the sequential dependence of matrix **Y** on matrix **X**.

By $\Gamma_{\pi_X}^S$ we mean a set of all the partitional sequential dependences of matrix **Y** on matrix **X**.

By notation $\Gamma_{\pi_X}^{Sd}$ we mean the set of all the expressions representing the sequential disjunctive dependence of matrix **Y** on matrix **X**. Correspondingly $\Gamma_{\pi_X}^{Sc}$ is the set of all the expressions representing the sequential conjunctive dependence of matrix **Y** on matrix **X**. Similarly as it was done for combinational systems we introduce the sequential systems like $\langle \mathbf{X}, \mathbf{Y}, \Omega^s \rangle$, $\langle \mathbf{X}, \mathbf{Y}, \Gamma_{\pi_X}^{Sd} \rangle$, $\langle \mathbf{X}, \mathbf{Y}, \Gamma_{\pi_X}^{Sc} \rangle$ and so on.

Let be given two sequential systems $\langle \mathbf{X}, \mathbf{Y}, \Gamma_{\pi_X}^{Sd} \rangle$ and $\langle V, Z, \Psi_{\tau_V}^{S} \rangle$ such that each block of partition π_X is included in some block of the partition τ_V.

Theorem 5

System $\langle \mathbf{X}, \mathbf{y}, \Gamma^{Sd}_{\pi x} \rangle$ is included in system $\langle V, \mathbf{z}, \Gamma^{S}_{\tau v} \rangle$ if and only if the following assumptions are satisfied.

1. There is one-to-one mapping h_1: $\mathbf{X} \to V$.

2. For each $y(j) \mp y(j-1)$ or $(y(j) = 1$ and $j = 1)$ there exists a minimum submatrix $\mathbf{X}_\pi$ such that $R_{X_\pi}(j) \mp R_{X_\pi}(k)$ whenever $y(k) \mp y(j)$.

3. There exists $z(1) \mp z(l-1)$ or $(z(1) = 1$ and $l = 1)$ and exists submatrix $V_\tau = h_1(\mathbf{X}_\pi)$ satisfying the following requirement $R_{X_\pi}(j) \mp R_{V_\tau}(l)$.

4. $R_{V_\tau}(l) \mp R_{V_\tau}(u)$ whenever $z(u) \mp z(l)$.

Proof

We have two special cases of assumption 2

(i) $y(j) = 1$ and $y(j-1) = 0$

(ii) $y(j) = 0$ and $y(j-1) = 1$

case (*i*) implies that vector $\mathbf{y}_1 = f^d_1(\mathbf{X})$, but case (*ii*) implies that $\mathbf{y}_2 = f^d_2(\mathbf{X})$ while $\mathbf{y} = \mathbf{y}_1 \bigcirc \mathbf{y}_2$. So it implies that the set $\Gamma^{Sd}_{\pi x}$ is not empty.

Assumption 1, 3 and 4 allow us to say that for each disjunctive form of sequential dependence $f^{Sd}_\pi(\mathbf{X}) \in \Gamma^{Sd}_{\pi x}$ there exists a form of sequential dependence $f^S_\tau(V) \in \Gamma^S_{\tau v}$ such that $f^{Sd}_\pi(\mathbf{X}) = h_1[f^S_\tau(V)]$. Therefore $\Gamma^{Sd}_{\pi x}$ is included in $\Gamma^{Sd}_{\pi v}$ so system $\langle \mathbf{X}, \mathbf{Y}, \Gamma^{Sd}_{\pi x} \rangle$ is included in system $\langle V, \mathbf{z}, \Gamma^{Sd}_{\tau v} \rangle$.

By some analogy we can construct a theorem allowing to state whether or not a given conjunctive sequential system is included in some other one, but constructing such theorem one must remember that then we have to find a minimum submatrix $\mathbf{X}_\pi$ for each $y(j)$ such that $\mathbf{R}_{X_\pi}(j) \mp R_{X_\pi}(l)$ whenever $y(l) \mp y(l-1)$ and $y(l) \mp y(j)$.

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