APPLICATION OF BENNETT WALL-MODEL IN ANALYSIS OF PULSE-WIDTH MODULATION PROCESSES

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Received Jul. 4, 1988

1. Introduction

Pulse-width modulation is a time modulation process with the pulse carrier wave. At pulse-time modulations, which are pulse equivalents of phase angle of harmonic carrier wave, take place linear dependence of time position of output signal edges on the modulating wave value. The magnitude and shape of pulses are not changed at such a process.

The modulating wave may vary the time of occurrence of the leading or the trailing edge of the output pulses. The message-bearing signal to be transmitted is composed of discrete values and each value must be uniquely defined by the width of a modulated pulse.

Pulse-width modulation is sometimes referred to in the literature on the subject [3], [4], [7] as pulse-length or pulse-duration modulation. The research of that time process was carried on with analytical methods mainly at the first half of our century. Recently, numerical method of looking into products of some kinds of pulse-width modulation processes with the use of fast Fourier transform algorithm has widely been adopted. It is particularly effective in consideration of difficulties which occur at the defining of analytical form of spectrum components [1], [2], [5], [7], [9], [10]. The paper is a probation of an application development of analytical methods in the testing of unexamining to date areas in pulse-width modulation processes. Results of that analysis can be used to modelling and optimalization of many deterministic dynamical systems with specified type of input functions in mechanics, electrical engineering, power engineering electronics and telecommunication carrying the analysis from time to frequency domain.

2. Area of the analysis

The analysis taken up in the work refers to an analytical consideration of sinusoidal pulse-width modulation processes with auxiliary carrier waves from the piecewise linear function class. Pulse-width modulation output signal can be categorized according to assumed forming method as a signal of modulation with the natural or regular sampling. For the case of regular sampling or so-called uniform sampling, pulse widths are proportional to message wave values at uniformly spaced sampling times and do not depend on modulating wave variation between the sampling times. For natural sampling, the output pulse widths are continuously depending on changes of the message signal values in the whole period of auxiliary wave.

At the open systems of sinusoidal pulse-width modulation with auxiliary waves harmonic modulating signal of the following shape is used: $\mu_M = A_M \sin(\omega_M t + \varphi_M)$, where A_M, ω_M and φ_M denote respectively magnitude, angular frequency and phase of the sinusoidal wave, and some piecewise linear signal. For would-be one-sided modulation at trailing edge modulated and leading edge fixed of output positive pulses the auxiliary wave we can describe as follows:

$$u_{a1} = (2A_a/T_a)t - A_a(2n+1) \wedge t \in [nT_a;\ (n+1)T_a);\ n \in \mathbf{N} \cup \{0\} \quad (2\text{-}1)$$

where A_a and T_a denote magnitude and period of the saw-tooth wave, respectively. For two-sided modulating process, the triangular auxiliary wave assumes the shape:

$$u_{a2} = (4A_a/T_a)(t - nT_a) - A_a \wedge t \in [nT_a;\ (n+1/2)T_a) \vee -(4A_a/T_a)[t - (n+1/2)T_a] + A_a \wedge t \in [(n+1/2)T_a);\ n \in \mathbf{N} \cup \{0\}\,. \quad (2\text{-}2)$$

Formation of output modulated pulses takes place during the comparing of the time curves (2-1) or (2-2) with sinusoidal function. Switching process of output wave occurs at the moment of equalling of both control signals, i.e. the modulating and auxiliary waves. So-called bipolar or two-level modulation takes place between two fixed values from the two-element set $D_2 \subset \mathbf{R} = \{E_1;\ E_2\}$ or for unipolar modulation process a would-be three-level modulation, too, between values from the set $D_3 \subset \mathbf{R} = \{E_1;\ 0;\ E_2\}$. Unipolar modulation depends on the polarity of output train, on polarity of the sinusoidal signal half-periods.

Among analytical methods, the Bennett wall-model was applied to date with a limited scale to examine mainly bipolar modulations with natural sampling [3], [4]. Adapted for conditions of regular sampling and three-level modulation conditions, the Bennett model will be presented. Modulation process with regular sampling of the harmonic wave may be examined without any restriction imposed on sampling frequency, instead, each modulation with natural sampling we can treat as a limiting case of the regular sampled modulation process.

3. Shapes of the Bennett wall-model

We can use in the analysis of harmonic spectrum of pulse-width modulation with auxiliary singnals the Fourier series expansion in two variables of an accessory periodical function $f(x, y)$ integrable in the Riemann sense in each regular area of domain given in the complex or real form, respectively:

$$f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm} e^{i(nx+my)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k_{nm}[\alpha_{nm} \cos nx \cos my +$$

$$+ \beta_{nm} \cos nx \sin my + \gamma_{nm} \sin nx \cos my + \varrho_{nm} \sin nx \sin my] \qquad (3\text{-}1)$$

where $c_{nn} = (1/4\Pi^2) \iint_Q f(x, y) e^{-i(nx+my)} d\sigma;\ n, m \in I \qquad (3\text{-}2)$

$$\alpha_{nm} = (1/\Pi^2) \iint_Q f(x, y) \cos nx \cos my \, d\sigma$$

$$\beta_{nm} = (1/\Pi^2) \iint_Q f(x, y) \cos nx \sin my \, d\sigma$$

$$\gamma_{nm} = (1/\Pi^2) \iint_Q f(x, y) \sin nx \cos my \, d\sigma$$

$$\varrho_{nm} = (1/\Pi^2) \iint_Q f(x, y) \sin nx \sin my \, d\sigma;\ n, m \in \mathbf{N} \cup \{0\} \qquad (3\text{-}3)$$

and k_{nm} assumes the following values: $1/4 \wedge n \wedge m = 0$, $1/2 \wedge n \vee m = 0$ or $1 \wedge n \wedge m \neq 0$, instead Q is a domain area for which $f(x, y)$ has the primary period in relation to x and y variables: $Q = \{(x, y) \in \mathbf{R} \times \mathbf{R}: x \in [-\Pi; \Pi) \wedge \wedge\, y \in [-\Pi; \Pi)\}$.

The shape of $f(x, y)$ function is defined by applying the Bennett model for considered type of modulation and assumes different forms depending on modulation process sort. So, for instance for bipolar one-sided modulation with regular sampling $f(x, y)$ assumes the shape as follows:

$$f_1(x, y) = E_1 \wedge (x, y) \in \{(x, y) \in \mathbf{R} \times \mathbf{R}:\ 2\Pi n \leq x < \Pi[1 + M \sin(y -$$

$$- (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M)] + 2\Pi n\} \vee E_2 \wedge (x, y) \in \{(x, y) \in \mathbf{R} \times \mathbf{R}:$$

$$\Pi[1 + M \sin(y - (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M)] + 2\Pi n \leq x < 2\Pi(n + 1)\} n \in I$$

$$(3\text{-}4)$$

where $M = A_M/A_a$ denotes depth of modulation, ω_a and ω_s pulsation of auxiliary and sampling wave respectively, instead ζ denotes so-called relative frequency of modulating and auxiliary signal. Fig. 1, illustrated the integration area of $f_1(x, y)$ shown with the function variability for following modulation parameter values: $\zeta = 18$, $M = 0.9$, $\varphi_M = -\Pi/6$ and $\omega_s = (1/6)\omega_a$.

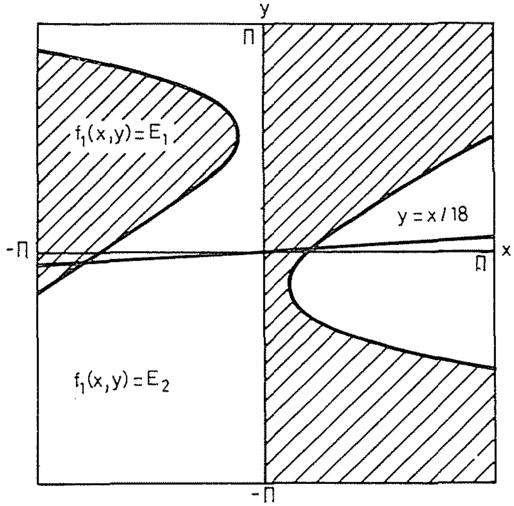

Fig. 1. Integration area for bipolar one-sided modulation

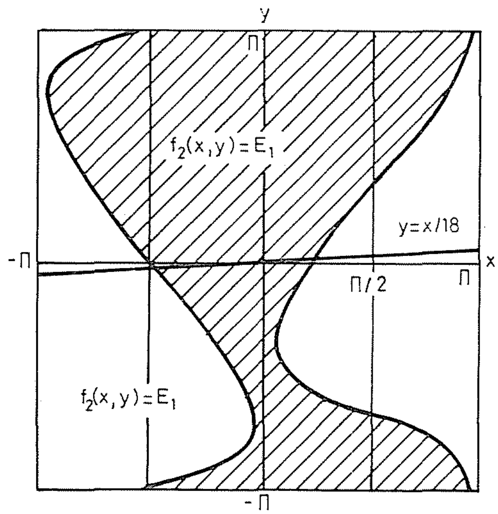

Fig. 2. Integration area for bipolar two-sided modulation

Form of $f(x, y)$ function for the following modulation processes: bipolar two-sided, unipolar one-sided and unipolar two-sided each with the regular sampling of harmonic signal are shown below by means of expressions (3-5), (3-6) and (3-7), instead Figs 2, 3 and 4, illustrate the integration area for given above modulation parameters and kinds of modulation processes.

$$f_2(x, y) = E_1 \wedge (x, y) \in \{(x, y) \in \mathbf{R} \times \mathbf{R}: -(\Pi/2)[1 + M \sin\left(y - (\omega_a/\zeta\omega_s)\cdot\right.$$

$$\left.\cdot(x - 2\Pi n) + \varphi_M\right)] + 2\Pi n \leq x < (\Pi/2)[1 + M \sin\left(y - (\omega_a/\zeta\omega_s)\cdot\right.$$

$$\cdot(x - 2\Pi n) + \varphi_M)] + 2\Pi n\} \vee E_2 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon (\Pi/2)\cdot$$
$$\cdot[1 + M \sin\left(y - (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M\right)] + 2\Pi n \leq x < (-\Pi/2)\cdot$$
$$\cdot[1 + M \sin\left(y - (\omega_a/\zeta\omega_s)(x - 2\Pi(n + 1)) + \varphi_M\right)] + 2\Pi(n + 1)\};\ n \in I \tag{3-5}$$

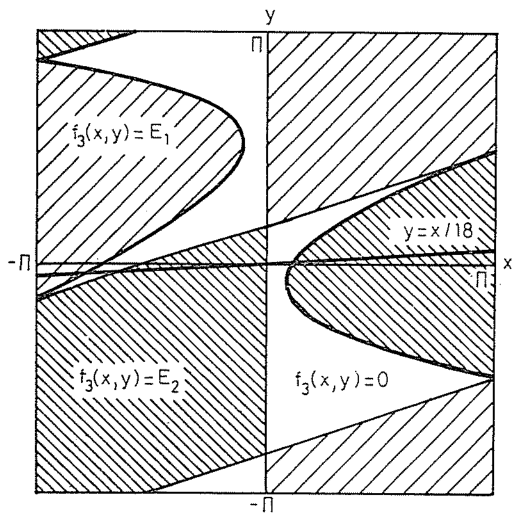

Fig. 3. Integration area for unipolar one-sided modulation

$$f_3(x, y) = E_1 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon 2\Pi n \leq x < \Pi[1 + M \sin\left(y -\right.$$
$$- (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M)] + 2\Pi n \wedge y \in [-\varphi_M + (\omega_a/\zeta\omega_s)(x -$$
$$- 2\Pi n) + 2\Pi m;\ -\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \Pi(2m + 1))\} \vee$$
$$\vee\ E_2 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon \Pi[1 + M \sin(y - (\omega_a/\zeta\omega_s)(x -$$
$$- 2\Pi n) + \varphi_M)] + 2\Pi n \leq x < 2\Pi(n + 1) \wedge y \in [-\varphi_M +$$
$$+ (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \Pi(2m + 1);\ -\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) +$$
$$+ 2\Pi(m + 1))\} \vee 0 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon \Pi[1 + M \sin\left(y -\right.$$
$$- (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M)] + 2\Pi n \leq x < 2\Pi(n + 1) \wedge$$
$$\wedge\ y \in [-\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + 2\Pi m;\ -\varphi_M + (\omega_a/\zeta\omega_s)(x -$$
$$- 2\Pi n) + \Pi(2m + 1)) \vee 2\Pi n \leq x < \Pi[1 + M \sin\left(y -\right.$$
$$- (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M)] + 2\Pi n \wedge y \in [-\varphi_M +$$
$$+ (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \Pi(2m + 1);\ -\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) +$$
$$+ 2\Pi(m + 1))\};\ m, n \in I \tag{3-6}$$

$$\begin{aligned}
f_4(x, y) = {} & E_1 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon (-\Pi/2)[1 + M \sin\big(y - \\
& - (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M\big)] + 2\Pi n \leq x < (\Pi/2)[1 + M \sin\big(y - \\
& - (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M\big)] + 2\Pi n \wedge y \in [-\varphi_M + (\omega_a/\zeta\omega_s)(x - \\
& - 2\Pi n) + 2\Pi m; -\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \Pi(2m + 1))\} \vee \\
& \vee E_2 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon (\Pi/2)[1 + M \sin\big(y - (\omega_a/\zeta\omega_s)(x - \\
& - 2\Pi n) + \varphi_M\big)] + 2\Pi n \leq x < (-\Pi/2)[1 + M \sin\big(y - \\
& - (\omega_a/\zeta\omega_s)(x - 2\Pi(n + 1)) + \varphi_M\big)] + 2\Pi(n + 1) \wedge y \in [-\varphi_M + \\
& + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \Pi(2m + 1); -\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \\
& + 2\Pi(m + 1))\} \vee 0 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon (\Pi/2)[1 + \\
& + M \sin\big(y - (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \varphi_M\big)] + 2\Pi n \leq x < (-\Pi/2)[1 + \\
& + M \sin\big(y - (\omega_a/\zeta\omega_s)(x - 2\Pi(n + 1)) + \varphi_M\big)] + 2\Pi(n + 1) \wedge \\
& \wedge y \in [-\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + 2\Pi m; -\varphi_M + (\omega_a/\zeta\omega_s)(x - \\
& - 2\Pi n) + \Pi(2m + 1)) \vee (-\Pi/2)[1 + M \sin\big(y - (\omega_a/\zeta\omega_s)(x - \\
& - 2\Pi n) + \varphi_M\big)] + 2\Pi n \leq x < (\Pi/2)[1 + M \sin\big(y - (\omega_a/\zeta\omega_s)(x - \\
& - 2\Pi n) + \varphi_M\big)] + 2\Pi n \wedge y \in [-\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + \\
& + \Pi(2m + 1); -\varphi_M + (\omega_a/\zeta\omega_s)(x - 2\Pi n) + 2\Pi(m + 1))\};\; m, n \in I\,.
\end{aligned} \tag{3-7}$$

Each of those functions is the Bennett 'wall'-model for suitable modulation process after comparing with the surface given by following parametric equation:

$$x = \omega_a t \wedge y = \omega_M t \tag{3-8}$$

the train of width modulated pulses. Application of the Bennett model adopted for any kind of sinusoidal pulse-width modulation with auxiliary carrier wave permits to obtain analytical form of otput signal spectrum. The direct looking into such a type of expansion is not possible in consideration of analytical uncertainty of output pulse switching point which is essential at the classical Fourier analysis in one time variable. So, the (3-1) expansion of one of the (3-4), (3-5), (3-6) or (3-7) functions with the coefficients given by (3-2) or (3-3) expressions and with regard to the (3-8) relationship gives harmonic spectrum of the appropriate modulation process in the form of double Fourier series. Modulations with natural sampling are respectively limiting cases for modulation processes with regular sampling. And so, for example, the bipolar one-sided modulation with natural sampling can be obtained from the bipolar

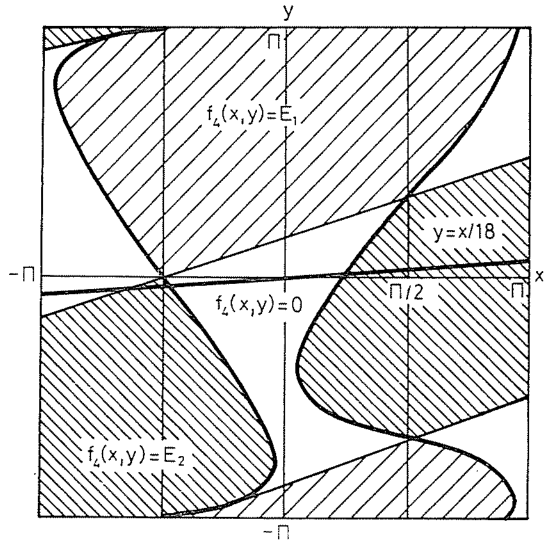

Fig. 4. Integration area for unipolar two-sided modulation

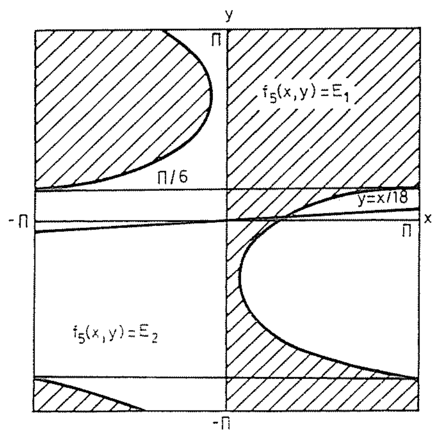

Fig. 5. Integration area for bipolar one-sided modulation with natural sampling

one-sided modulation process with regular sampling on the assumption that ω_s approaches infinity. Appropriate shape of accessory $f(x, y)$ function of the Bennett model and the function variability for that kind of process and for given above modulation parameters one may illustrate as follows:

$$f_5(x, y) = E_1 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon 2\Pi n \leq x < \Pi[1 + M \sin (y + \varphi_M)] + 2\Pi n\} \vee E_2 \wedge (x, y) \in \{(x, y) \in \mathbf{R}\times\mathbf{R}\colon \Pi[1 + M \sin (y + \varphi_M)] + 2\Pi n \leq x < 2\Pi(n + 1)\};\ n \in I \tag{3-9}$$

We can notice that curves defining the $f_5(x, y)$ function variability areas are non-distorted sinusoids depending thereby on the 'natural' way widths of output pulses.

4. Conclusion

The Bennett wall-model adopted for the miscellaneous processes of sinusoidal pulse-width modulation with auxiliary waves has been presented in the paper. To date, for instance, in the works [1, [2], [3], [4], [5], [8], one makes capital of the simplified Bennett models in a limited range to obtain generated harmonic spectrum by some types of modulation mainly at the cancellating assumptions as follows: $\varphi_M = 0$ and $\omega_s = \omega_a$. That does not permit us a look to the influence of the sinusoidal signal phase and unrestricted sampling frequency on the mechanisms of spectrum formation. Numerical methods of Fourier analysis were used to date at the outmost for processes with regular sampling defined only for sampling frequency equal to the auxiliary wave frequency or its doubled value. That permitted to differentiate so-called symmetric and asymmetric regular sampled modulation.

The proposed analytical method is an alternative for numerical fast Fourier transform allowing by the simple way to pull out extremes of an assumed objective function optimizing used modulation process or to utilize the resulted splitting at the analysis of transient states of dynamic objects exposed to such a type of input.

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