ANALYTIC DETERMINATION OF THE REFERENCE TRIHEDRON POSITION RELATED TO THE PRINCIPAL PROJECTION PLANE AND OF THE PROJECTION DIRECTION OF AN OBLIQUE AXONOMETRIC PROJECTION FROM GIVEN AXIS AND VARIATION COEFFICIENTS

J. MUNIOZGUREN, E. URANGA and C. GRAFULLA

Universidad del Pais Vasco — Escuela Superior de Ingenieros. Bilbao, Spain

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Introduction

The elaboration of an axonometric image from a mathematical model is usually laborious. This hard drafting work is quite justified when designers or researchers desire to be quickly understood. Receivers of these images can be laymen in communicated matters as well as professionals of them.

The usual axonometry allows the selection of no more than one of the variation coefficients.

Whether arbitrary selection of variation coefficients is required, the axonometry is oblique as dated by K. POHLKE in his theorem and demonstrated by H. A. SCHWARZ. Thus, sketching with three axes and three arbitrary variation coefficients is feasible.

Furthermore, once three axes and the value of three variation coefficients are stated, it is often interesting (or needed) to determine the drawing plane position related to the reference trihedron, as well as the projection direction.

The aim of this communication is to show the analytic determination of the parameters cited above as well as to report some studied cases.

Development

Due to experience, it is accepted as true that an image in oblique axonometry presents a real aspect. The mathematical justification is given by the fundamental theorem of the oblique axonometry, stated by K. POHLKE. Up to now, several demonstrations have been given. (The first by H. A. SCHARZ).

The theorem is reported as follows: "A Tri-rectangular trihedron can be projected cylindrically on a plane, in order that the axes form — among each other any angle and the variation coefficients are arbitrary."

To act within a definite oblique axonometric system, in the free way the mentioned theorem permits, it is necessary to state the relative position between the "drawing plane" and "reference trihedron", as well as projection oblique direction, as prior step to the implementation in a computerized sketch system.

The first step is to project the trihedron axes on the drawing plane, as well as to define the variation coefficients of each of them.

Being R, S, T the projections of the tri-rectangular trihedron U, V, W on the plane Π, according to direction $\bar{X}$, knowing the mentioned projections and the plane Π, the components of the unitarian vectors $\overline{\mathbf{U}}, \overline{\mathbf{V}}, \overline{\mathbf{W}}$ and $\overline{\mathbf{X}}$ are to be attained Fig. 1.

An orthonormade reference is chosen $\bar{i}, \bar{j}, \bar{k}$, so that the horizontal plan XOY coincides with the plane Π, the projection R is on the axis OX and the origin 0 of the trihedron $OUVW$ coincides with that of the reference. In these conditions the relations between the vectors are (Fig. 2):

$$\overline{\mathbf{U}} = \overline{\mathbf{R}} + \overline{\mathbf{X}}$$

$$\overline{\mathbf{V}} = \overline{\mathbf{S}} + \lambda\overline{\mathbf{X}}$$

$$\overline{\mathbf{W}} = \overline{\mathbf{T}} + \mu\overline{\mathbf{X}}.$$

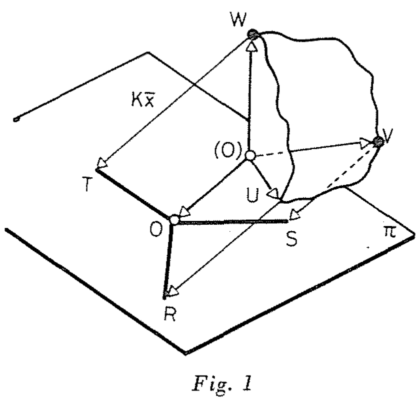

Fig. 1

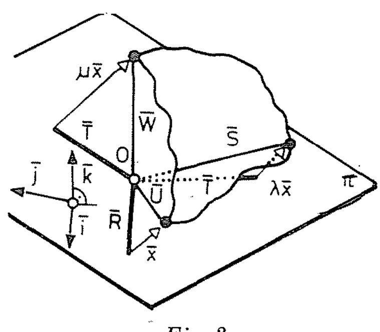

Fig. 2

Where $\overline{\mathbf{U}}, \overline{\mathbf{V}}, \overline{\mathbf{W}}, \overline{\mathbf{X}}, \lambda, \mu$, are unknown and $\bar{R}, \bar{S}, \bar{T}$, are known:

$$\overline{\mathbf{R}} = r\bar{\mathbf{i}} + o\bar{\mathbf{j}} + o\overline{\mathbf{K}}$$
$$\bar{\mathbf{S}} = s_1\bar{\mathbf{i}} + s_2\bar{\mathbf{j}} + o\overline{\mathbf{K}}$$
$$\overline{\mathbf{T}} = t_1\bar{\mathbf{i}} + t_2\bar{\mathbf{j}} + o\overline{\mathbf{K}}.$$

As $\mathbf{U}, \mathbf{V}, \mathbf{W}$ form a trihedron, it must be accomplished:

$$\begin{cases} \overline{\mathbf{U}} \times \overline{\mathbf{V}} = \overline{\mathbf{W}} \\ \overline{\mathbf{V}} \times \overline{\mathbf{W}} = \overline{\mathbf{U}} \qquad \text{in consequence} \\ \overline{\mathbf{W}} \times \overline{\mathbf{U}} = \overline{\mathbf{V}} \end{cases}$$

$$\begin{cases} (\overline{\mathbf{R}} + \overline{\mathbf{X}}) \times (\bar{\mathbf{S}} + \lambda\overline{\mathbf{X}}) = (\overline{\mathbf{T}} + \mu\overline{\mathbf{X}}) \\ (\bar{\mathbf{S}} + \lambda\overline{\mathbf{X}}) \times (\overline{\mathbf{T}} + \mu\overline{\mathbf{X}}) = (\overline{\mathbf{R}} + \overline{\mathbf{X}}) \\ (\overline{\mathbf{T}} + \mu\overline{\mathbf{X}}) \times (\overline{\mathbf{R}} + \overline{\mathbf{X}}) = (\bar{\mathbf{S}} + \lambda\overline{\mathbf{X}}). \end{cases}$$

Acting on vector products, the result is:

$$\begin{cases} \overline{\mathbf{R}} \times \bar{\mathbf{S}} + \overline{\mathbf{R}} \times (\lambda\overline{\mathbf{X}}) + \overline{\mathbf{X}} \times \bar{\mathbf{S}} = \overline{\mathbf{T}} + (\mu\overline{\mathbf{X}}) & (1) \\ \bar{\mathbf{S}} \times \overline{\mathbf{T}} + \bar{\mathbf{S}} \times (\mu\overline{\mathbf{X}}) + (\lambda\overline{\mathbf{X}}) \times \overline{\mathbf{T}} = \overline{\mathbf{R}} + \overline{\mathbf{X}} & (2) \\ \overline{\mathbf{T}} \times \overline{\mathbf{R}} + \overline{\mathbf{T}} \times \overline{\mathbf{X}} + (\mu\overline{\mathbf{X}}) \times \overline{\mathbf{R}} = \bar{\mathbf{S}} + (\lambda\overline{\mathbf{X}}). & (3) \end{cases}$$

Acting on expressions (1), (2), (3), the following results are obtained:

$$\begin{cases} \mu x_1 = -s_2x_3 - t_1 \\ \mu x_2 = s_1x_3 - r\lambda x_3 - t_2 \\ \mu x_3 = rs_2 + x_1s_2 - x_2s_1 + r\lambda x_2 \end{cases}$$

$$\begin{cases} \lambda x_1 = t_2x_3 - s_1 \\ \lambda x_2 = -t_1x_3 + r\mu x_3 - s_2 \\ \lambda x_3 = -rt_2 + t_1x_2 - t_2x_1 - r\mu x_2 \end{cases}$$

$$\begin{cases} r + x_1 = s_2\mu x_3 - t_2\lambda x_3 \\ \quad x_2 = -s_1\mu x_3 + t_1\lambda x_3 \\ \quad x_3 = s_1t_2 - s_2t_1 + s_1\mu x_2 - s_2\mu x_1 + t_2\lambda x_1 - t_1\lambda x_2 \end{cases}$$

$$\begin{cases} \mu x_1 = -s_2x_3 - t_1 \\ \mu x_2 = (s_1x_3 - r^2t_2 - rt_1x_2 + rt_2x_1 - t_2)\dfrac{1}{1 - r^2} \\ \mu x_3 = (s_2x_1 - s_1x_2 - rt_1x_3)\dfrac{1}{1 - r^2} \end{cases}$$

$$\lambda x_1 = t_2 x_3 - s_1$$

$$\lambda x_2 = (-t_1 x_3 + r^2 s_2 + r s_2 x_1 - r s_1 x_2 - s_2)\frac{1}{1-r^2}$$

$$\lambda x_3 = (t_1 x_2 - t_2 x_1 - r s_1 x_3)\frac{1}{1-r^2}.$$

Finally the following matritial relation is obtained:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{s_2^2 + t_2^2}{1-r^2} - 1 & \dfrac{-s_1 s_2 - t_1 t_2}{1-r^2} & \dfrac{r s_1 t_2 - r s_2 t_1}{1-r^2} \\ \dfrac{-s_1 s_2 - t_1 t_2}{1-r^2} & \dfrac{s_1^2 + t_1^2}{1-r^2} - 1 & 0 \\ \dfrac{r s_1 t_2 - r s_2 t_1}{1-r^2} & 0 & \dfrac{s_1^2 + t_1^2}{1-r^2} + s_2^2 + t_2^2 - 1 \end{Bmatrix}^{-1} \begin{Bmatrix} r \\ 0 \\ [s_1 t_2 - s_2 t_1] \end{Bmatrix}$$

The relative position of the trihedron in respect to the projection plane, as well as its directions are found.

Application

The procedure to apply in practice the POHLKE Theorem to the geometric model is as follows:

1) We are to determine the axonometric axes projections on the drawing plane and the variation coefficients, as long as the aspect of the perspective is the suitable one (using *unit regular hexahedron*).

2) Based on the axes and coefficients established in the first step, the projection direction and the relative position between the drawing plane and the referenced trihedron of the axonometry, is to be calculated, (see item 1).

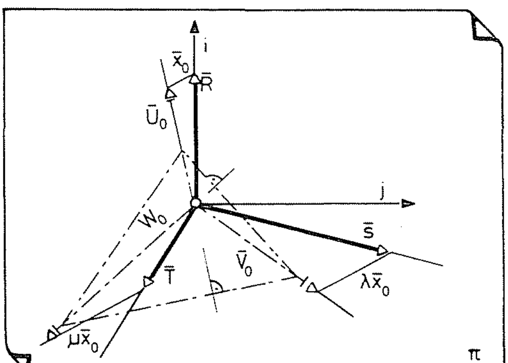

Fig. 3

The projection direction is:

$$\overline{\mathbf{X}} = X_1\bar{\mathbf{i}} + X_2\bar{\mathbf{J}} + X_3\overline{\mathbf{K}}.$$

The orthogonal projections of the axis of the axonometric system are determined on the drawing plane: $\overline{\mathbf{U}}_0, \overline{\mathbf{V}}_0, \overline{\mathbf{W}}_0$, by the orthogonal projection $\overline{\mathbf{X}}_0$ of the projection direction. A fundamental triangle is marked on — these, the heights being the orthogonal axonometric axes.

Lastly, in the direction $\overline{\mathbf{X}}_0$, the sketch triangle on the oblique axes is obtained.

Once the projection of a fundamental triangle is based on the axes of the oblique axonometry, any graphic operation can be solved rather easily, by debasing the co-ordinated planes.

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J. MUNIOZGUREN, E. URANGA, C. GRAFULLA
Alda Urquijo s/n.
48013—Bilbao
España