PLANE INTERPRETATION OF LUMINOUS RADIATIONS

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Introduction

The present work is a follow-up of the one published (pages 222 through 238) in the volume for the International Congress on Linear Algebra and its Applications, held in September 1983 in Vitoria (Basque Country).

Let us sum up, in principle, some of the results obtained at the above-mentioned Congress:

1. It was shown that the set of radiations from the Electromagnetic Spectrum (both the visible and the invisible ones for the human eye) had a structure **R**-vectorial space of dimension 3, and therefore isomorphous with the vectorial space $\mathbf{R}^3$. Every radiations from the Spectrum could be synthesized by three colour lights: red, green and violet blue, with light quantities R, V and A respectively, and were defined by the tern (R, V, A). If $R + V + A > 0$, the radiation was visible to the human eye; should $R + V + A \leq 0$, radiation was said invisible.

2. Let us remember as well the following definition:

The trilinear coordinates X, Y, Z of a point **P** are the distances d_1, d_2, d_3 from point **P** to the sides of a triangle (taken as a reference), multiplied by three parameters μ_1, μ_2, μ_3. So,

$$X = \mu_1, d_1, \ \ Y = \mu_2, d_2, \ \ Z = \mu_3, d_3.$$

After selecting the triangle in Figure 1 (named ideal reference triangle), and for μ_1, μ_2, μ_3 values $1, 1, \sqrt{2}$ respectively, we have, by simple calculation, the tying ratio

$$X + Y + Z = 1$$

As the set of visible radiations or colours is

$$\mathbf{M} = \{(R, V, A) \in \mathbf{R}^3 / R + V + A > 0\}$$

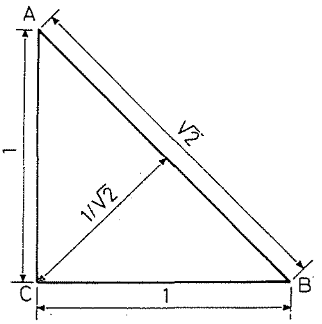

Fig. 1

and as the terns $(kR, kV, kA) \in \mathbf{M}$, $k > 0$, which have proportional coordinates, represent the same colour C, nothing prevents us from taking as representation for them all the term (r, v, a) shown by

$$r = \frac{kR}{k(R+V+A)}, \quad v = \frac{kV}{k(R+V+A)}, \quad a = \frac{kA}{k(R+V+A)}$$

which also clearly expresses C, and where it is verified

$$r + v + a = 1$$

which, as can be seen, is precisely the tying ratio among the trilinear coordinates. Therefore, and in order to represent the colours, these coordinates could be used, throughout in the previously stated reference.

After finishing this summary, we begin now our work with some considerations on trilinear coordinates.

1. The step to pass from the trilinear system to the Cartesian orthonormade

Supposing that the equations of the triangle's verges of Figure 2 are given by

$$CB \equiv x \cos(\alpha_1) + y \sin(\alpha_1) - p_1 = 0$$
$$BA \equiv x \cos(\alpha_2) + y \sin(\alpha_2) - p_2 = 0$$
$$AC \equiv x \cos(\alpha_3) + y \sin(\alpha_3) - p_3 = 0$$

and as the distances from point $\mathbf{P}(x, y)$ to such verges, are proportionals to

$$x \cos(\alpha_1) + y \sin(\alpha_1) - p_1$$
$$x \cos(\alpha_2) + y \sin(\alpha_2) - p_2$$
$$x \cos(\alpha_1) + y \sin(\alpha_3) - p_3$$

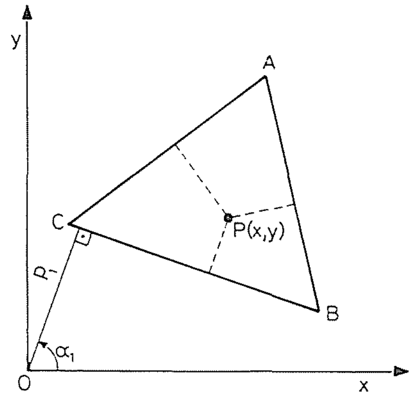

Fig. 2

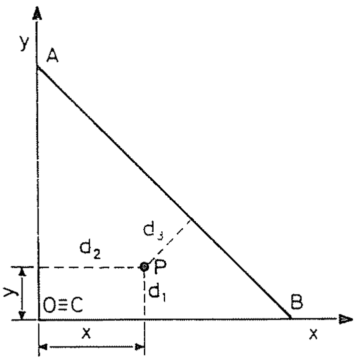

Fig. 3

we shall have that the relations which join the trilinear coordinates X, Y, Z, of **P** with the cartesian x, y, will be

$$X = \mu_1 k_1[x \cos(\alpha_1) + y \sin(\alpha_1) - p_1]$$

$$Y = \mu_2 k_2[x \cos(\alpha_2) + y \sin(\alpha_2) - p_2]$$

$$Z = \mu_3 k_3[x \cos(\alpha_3) + y \sin(\alpha_3) - p_3]$$

from what k_1, k_2, k_3 differ from the reference triangle.

Because the reference triangle which we have chosen is the one from Figure 3 (ideal reference triangle), we are able to write

$$X = \mu_1 d_1 = 1 * y$$

$$Y = \mu_2 d_2 = 1 * x$$

$$Z = \mu_3 d_3 = \sqrt{2} * d_3 .$$

Now k_1, k_2, k_3 are direct, because $X + Y + Z = 1$, and it is necessary that

$$\begin{aligned} X &= y \\ Y &= x \\ Z &= 1 - x - y. \end{aligned} \tag{1}$$

2. Equation of the straight line by the trilinear coordinates

Being $mx + ny + p = 0$ the equation of the straight line on the plane, and substituting in the equation the relations (1), we have

$$mY + nX + p = 0$$

and because $X + Y + Z = 1$, we can say that

$$mY + nX + p(X + Y + Z) = 0$$

in such way the trilinear coordinates, the equation of the straight line will be given by the homogeneous expression

$$aX + bY + cZ = 0$$

(expression which is in general independent from the reference triangle to be taken).

A) Equation of a straight line in relation with the distances to the verges of the reference triangle

Beginning with the general equation $aX + bY + cZ = 0$ and examining Figure 4, we shall have

$$\begin{aligned} X_D &= \mu_1 d_1 = \ 1 * \overline{CD} = \overline{CD} \\ Y_D &= \mu_2 d_2 = \ 1 * 0 = 0 \\ Z_D &= \mu_3 d_3 = \sqrt{2} * [-\overline{AD} \sin(\pi/4)] = -\overline{AD} \end{aligned}$$

because the point **D** belongs to the straight line

$$a * \overline{CD} + b * 0 - c * \overline{AD} = 0 \Rightarrow \frac{a}{\overline{AD}} = \frac{c}{\overline{CD}} \tag{2}$$

and because of the resemblance of the triangles $APD\triangle$ and $CQD\triangle$, results

$$\frac{\overline{AD}}{p} = \frac{\overline{CD}}{r} \overset{(2)}{\Rightarrow} \frac{a}{p} = \frac{c}{r}$$

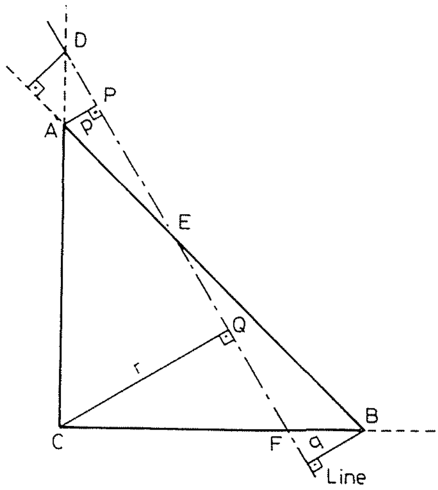

Fig. 4

from the points "E" and "F" we shall have in the same way

$$\frac{a}{p} = \frac{b}{q}, \quad \frac{b}{q} = \frac{c}{r}$$

so that

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$$

from all of those results the next equation

$$pX + qY + rZ = 0$$

or which is the same

$$\frac{p}{r}X + \frac{q}{r}Y + Z = 0.$$

If the straight line gets further indefinitely, the rates (p/r) and (q/r) for being quotients of infinite equivalents tend to 1; so the equation

$$X + Y + Z = 0$$

represents the straight line of the infinite of the plane considered.

B) Trilinear equation of a straight line that passes by two points

If $\mathbf{P}_1(X_1, Y_1, Z_1)$ and $\mathbf{P}_2(X_2, Y_2, Z_2)$ are those two points, we shall have

$$aX + bY + cZ = 0$$
$$aX_1 + bY_1 + cZ_1 = 0$$
$$aX_2 + bY_2 + cZ_2 = 0$$

which is a homogeneous system with three equations and three hidden reasons a, b, c, not simultaneously void, so

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0$$

will be the equation of the straight line.

3. Intersection point of two trilinear straight lines

Let the straight lines

$$a_1X + b_1Y + c_1Z = 0$$

$$a_2X + b_2Y + c_2Z = 0.$$

So as to get their intersection point, it is clear that the system should be determined

$$\begin{aligned} a_1X + b_1Y + c_1Z &= 0 \\ a_2X + b_2Y + c_2Z &= 0 \\ X + Y + Z &= 1 \end{aligned}$$

so

$$X = \frac{\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}}{A}, \qquad Y = \frac{\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ 1 & 1 & 1 \end{vmatrix}}{A}, \qquad Z = \frac{\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{A}$$

where

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}$$

or the same thing

$$X = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{A}, \qquad Y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{A}, \qquad Z = \frac{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}{A}.$$

Employing the last knowledge and with easy computations not presented here, the results of Figure 5 can be obtained (the different straight lines have been got by the concept of "a straight line that passes by two points").

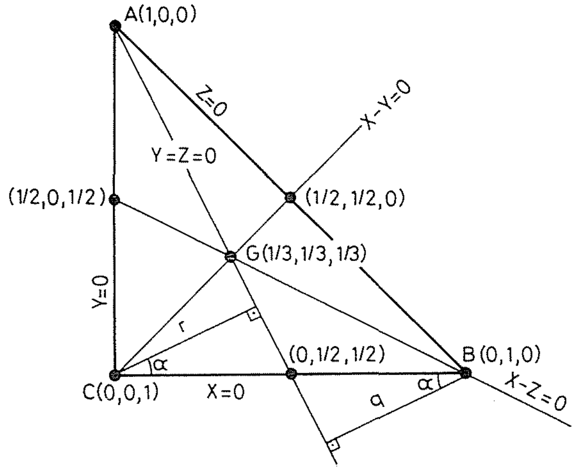

Fig. 5

The barycenter **G**(1/3, 1/3, 1/3) of the ideal triangle of reference represents the colour "white"; the orthocenter **C**(0, 0, 1) represents the colour "violet blue"; the other vertexes **A**(1, 0, 0) and **B**(0, 1, 0) represent the colours "red" and "green".

4. Trilinear distance from a point to a straight line

Let the straight line be $aX + bY + cZ = 0$ and the point $\mathbf{P}(X_1, Y_1, Z_1)$. As in the Cartesian system, the distance will be given by the expression

$$d = \frac{|mx_1 + ny_1 + p|}{\sqrt{m^2 + n^2}}, \quad r \equiv mx + ny + p = 0$$

in the trilinear system we shall have

$$\begin{gathered} aX + bY + cZ = ay + bx + c(1 - x - y) \Rightarrow \\ \Rightarrow r \equiv (b - c)x + (a - c)y + c = 0 \qquad (3) \\ d = \frac{|(b - c)x_1 + (a - c)y_1 + c|}{\sqrt{(b - c)^2 + (a - c)^2}} = \\ = \frac{|(b - c)Y_1 + (a - c)X_1 + c(X_1 + Y_1 + Z_1)|}{\sqrt{(b - c)^2 + (a - c)^2}} = \\ = \frac{|aX_1 + bY_1 + cZ_1|}{\sqrt{(b - c)^2 + (a - c)^2}}. \end{gathered}$$

It can immediately be seen that the distance between two points $\mathbf{P}(X_1, Y_1, Z_1)$ and $\mathbf{P}(X_2, Y_2, Z_2)$, can be expressed by

$$d = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}.$$

5. Angle of two straight lines in trilinear coordinates

Let the straight lines be

$$a_1X + b_1Y + c_1Z = 0$$

$$a_2X + b_2Y + c_2Z = 0$$

with cartesian equations

$$m_1x + n_1y + p_1 = 0$$

$$m_2x + n_2y + p_2 = 0.$$

Knowing that

$$\mathrm{tg}\,(\alpha) = \frac{m_1n_2 + m_2n_1}{n_1n_2 + m_1m_2}$$

and considering (3), results

$$\mathrm{tg}\,(\alpha) = \frac{(b_1 - c_1)(a_2 - c_2) + (b_2 - c_2)(a_1 - c_1)}{(a_1 - c_1)(a_2 - c_2) - (b_1 - c_1)(b_2 - c_2)}\,.$$

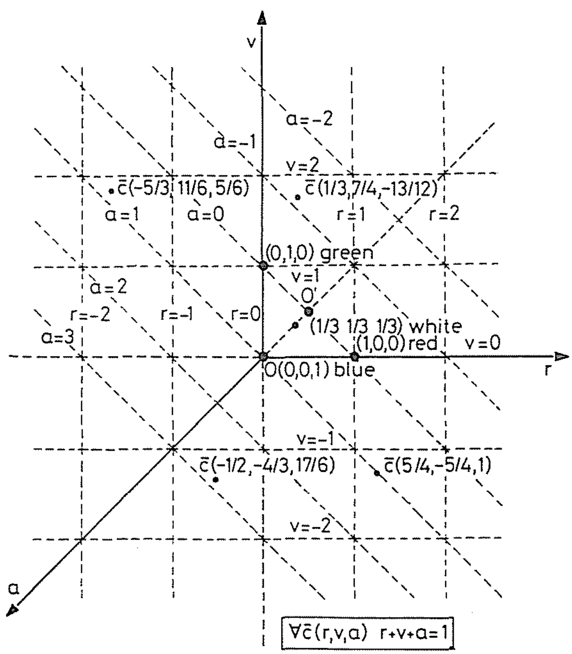

Fig. 6

Application of trilinear coordinates to the representation of colours on a plane

All the above stated is graphically shown in Figure 6, where we decided to draw three axis r, v, a, which greatly help our obtaining the three coordinates of any colour.

In Figure 6, the point **O**(0, 0, 1) is the origin of axis r and v, which measure the quantities of red and green light which make up a specific colour; the point **O**'(1/2, 1/2, 0) being the origin of axis a, which axis expresses the quantity of violet blue light.

Notice, finally, that our ideal reference triangle has red (1, 0, 0), green (0, 1, 0) and biolet blue (0, 0, 1) colours as vertex.

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