ON THE STRUCTURE OF LARGE COMPUTER SYSTEMS

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Introduction

Let C be a large set of computers, on which we define a "$\leqq$" partial ordering. If $a, b \in C$, then $a \leqq b$ if a is a submachine of b. Let P be a — non void — set of problems. If $p \in P$, then we define a $\varrho(p)$ relation on the following way, if $a, b \in C$, then $a\varrho(p)b$ if the problem p can be solved by a and b.

Now consider two "ideal" elements $I \neq o$ and the set $C \cup I \cup o$ on which we define a $\wedge$ and a $\vee$ operation as follows:

If $a, b \in C$, then

$$a \wedge b = \begin{cases} a \cap b & \text{if} \quad a \cap b \in C, \\ o & \text{if} \quad a \cap b \notin C, \end{cases} \tag{1}$$

$$a \wedge o = o \wedge a = o, \; a \wedge I = I \wedge a = a, \; o \wedge o = o, \; I \wedge I = I \tag{2}$$

$$o \wedge I = I \wedge o = o$$

and

$$a \vee b = \begin{cases} a \cup b & \text{if} \quad a \cup b \in C, \\ I & \text{if} \quad a \cup b \notin C, \end{cases} \tag{3}$$

$$a \wedge o = o \vee a = a, \; a \vee I = I \vee a = I, \; o \vee o = o, \; I \vee I = I, \tag{4}$$

$$o \vee I = I \vee o = I.$$

Now consider the set P and we suppose that if $p, q \in P$, then $p \cup q \in P$. Let i be an "ideal" element and on $P \cup i$ we define a $\sqcap$ and a $\sqcup$ operation as follows:

If $p, q \in P$, then

$$p \sqcap q = \begin{cases} p \cap q & \text{if} \quad p \cap q \in P, \\ i & \text{if} \quad p \cap q \notin P, \end{cases} \tag{5}$$

$$p \sqcap i = i \sqcap p = i, \; i \sqcap i = i, \tag{6}$$

and

$$p \sqcup q = p \cup q, \; p \sqcup i = p, \; i \sqcup i = i. \tag{7}$$

If $\alpha \in C$ then we define a $\sigma(a)$ relation on the following way, if $p, q \in P$, then $p\sigma(a)q$ if the problems p, q can be solved by a.

In this lecture the basic properties of the relational structures $\big(C \cup o \cup I, \wedge, \vee, \varrho(p)\big)$ and $(P \cup i, \sqcap, \sqcup, \sigma(a))$; $(p \in P, a \in C)$ are investigated.

The results of this work have many useful applications for parallel and distributed computation.

For the study of the field treated in this lecture, the reader is referred to the works [1], [2], [3], [4].

On the structure $(C \cup o \cup I, \wedge, \vee)$

It is easy to see that $(C \cup o \cup I, \wedge)$ and $(C \cup o \cup I, \vee)$ are idempotent groupoids but in generally are not semilattices.

It may be formulated a question: if for a set C the triplet $(C \cup o \cup I, \wedge, \vee)$ is a lattice, what consequences are concerning the set C.

The answer is the following theorem [4]:

Theorem 1. The elements of the set $C \cup o \cup I$ under the operations $\wedge, \vee$ form a lattice if and only if for any triplet $a, b, c \in C$, the following two conditions hold:

1°. $a \cap b \cap c \notin C$ or if $a \cap b \cap c \in C$, then either both $a \cap b$ and $b \cap c \in \in C$, or none of them belongs to C,

2°. $a \cup b \cup c \notin C$ or if $a \cup b \cup c \in C$, then either both $a \cup b$ and $b \cup c \in \in H$, or none of them belongs to C.

We can see that this theorem gives an exact limit for the applications of lattice theory in the structure analysis of large computer systems.

If $p \in P$, then denote by $C(p)$ the set of all a elements of C for which the problem p can be solved by a.

We suppose that $C(i) = C$, and if $p \in P$, then there is at least one a element of C for which p can be solved by a, and if $b \in C$, then there is at least one q element of P which can be solved by b.

It is easy to see that if $p, q \in P$ and $p \subseteq q$, then $C(p) \supseteqq C(q)$. If $p \cap q \in P$, then $C(p \cap q) \supseteq C(p) \cup C(q)$, and if

$$p, q \in P$$

then $C(p \cup q) = C(p) \cap C(q)$, which is not empty, because we supposed that $p \cup q \in P$. Denote by $\widehat{C}$ the set of all $C(p), (p \in P)$ sets.

Now consider the structure $(\widehat{C} \cup C, \cap)$.

If $p, q \in P$, then $C(p) \cap C(q) = C(p \cup q) \in \widehat{C}$ and $C(p) \cap C = C(p) \in \widehat{C}$, further $C \cap C = C \in \widehat{C} \cup C$.

If $A, B, C \in \hat{C} \cup C$, then

$$(A \cap B) \cap C = A \cap (B \cap C), \tag{8}$$

$$A \cap B = B \cap A \tag{9}$$

$$A \cap A = A. \tag{10}$$

Therefore $(\hat{C} \cup C, \cap)$ is a semilattice, where $C \cap A = A$.
Now on $C \cup C$ we define a $\tilde{\cup}$ operation as follows: if $p, q \in P$, then

$$C(p) \tilde{\cup} C(q) = \begin{cases} C(p \cap q) & \text{if} \quad p \cap q \in P, \\ C & \text{if} \quad p \cap q \notin P, \end{cases} \tag{11}$$

and

$$C(p) \tilde{\cup} C = C \tilde{\cup} C(p) = C, \; C \tilde{\cup} C = C. \tag{12}$$

We can see that if $p, q \in P$, then

$$C(p) \tilde{\cup} C(q) = C(q) \tilde{\cup} C(p), \tag{13}$$

and

$$C(p) \tilde{\cup} C(p) = C(p) \tag{14}$$

further

$$\big(C(p) \cap C(q)\big) \tilde{\cup} C(p) = C(p \cup q) \tilde{\cup} C(p) = C(p) \tag{15}$$

and

$$\big(C(p) \tilde{\cup} C(q)\big) \cap C(p) = \begin{cases} C(p \cap q) \cap C(p) = C(p) & \text{if} \quad p \cap q \in P, \\ C \cap C(p) = C(p) & \text{if} \quad p \cap q \notin P, \end{cases} = C(p). \tag{16}$$

If $A, B, C \in \hat{C} \cup C$ and $C \in \{A, B, C\}$, then

$$(A \tilde{\cup} B) \tilde{\cup} C = A \tilde{\cup} (B \tilde{\cup} C). \tag{17}$$

Next we shall use the following notations:

$$(p, q) r = (C(p) \tilde{\cup} C(q)) \tilde{\cup} C(r), \tag{18}$$

$$p(q, r) = C(p) \tilde{\cup} (C(q) \tilde{\cup} C(r)), \tag{19}$$

where $p, q, r \in P$.

If $p \cap q \cap r$, $p \cap q$, $q \cap r \in P$, then

$$(p, q) r = C(p \cap q \cap r), \tag{20}$$

and

$$p(q, r) = C(p \cap q \cap r). \tag{21}$$

If $p \cap q$, $p \cap q \cap r \in P$ and $q \cap r \notin P$, then

$$(p, q)r = C(p \cap q \cap r) \tag{22}$$

and

$$p(q, r) = C(p) \tilde{\cup} C = C. \tag{23}$$

If $q \cap r$, $p \cap q \cap r \in P$ and $p \cap q \notin P$, then

$$(p, q)r = C \tilde{\cup} C(r) = C, \tag{24}$$

and

$$p(q, r) = C(p \cap q \cap r). \tag{25}$$

If $p \cap q \cap r \in P$ and $p \cap q, q \cap r \notin P$ then

$$(p, q)r = C \quad (26) \text{ and } \quad p(q, r) = C. \tag{27}$$

If $p \cap q \cap r \notin P$ and $p \cap q, q \cap r \in P$, then

$$(p, q)r = C \quad (28) \text{ and } \quad p(q, r) = C. \tag{29}$$

If $p \cap q, p \cap q \cap r \notin P$ and $q \cap r \in P$, then

$$(p, q)r = C \quad (30) \text{ and } \quad p(q, r) = C. \tag{31}$$

If $q \cap r, p \cap q \cap r \notin P$ and $p \cap q \in P$, then

$$(p, q)r = C \quad (32) \text{ and } \quad p(q, r) = C. \tag{33}$$

If $p \cap q \cap r \notin P$ and $p \cap q, q \cap r \notin P$, then

$$(p, q)r = C \quad \text{and } (34) \quad p(q, r) = C. \tag{35}$$

Therefore we have to prove the following theorem:

Theorem 2. The elements of the set $\hat{C} \cup C$ under the operations $\cap$, $\tilde{\cup}$ form a lattice if and only if for any triplet $p, q, r \in P$ it holds the following condition:

$p \cap q \cap r \notin P$, or if $p \cap q \cap r \in P$, then either both $p \cap q$ and $q \cap r$ belong to P, or none of them belongs to P.

Next we consider some simple but important basic properties of the relations $\varrho(p)$, $(p \in P)$ and $\leqq$.

It is easy to see that if $a, b, c \in C$ and $p, q \in P$, then

$$a\varrho(p)b \Leftrightarrow a, b \in C(p), \tag{36}$$

if $a, b \in C(p)$ or $a, b \in C(p)$, then $a\big(\varrho(p) + \varrho(q)\big)b$, further if $a, c \in C(p)$ and $b, c \in C(p)$, then

$$a(\varrho(p)\varrho(q))\, b. \tag{37}$$

Now we suppose that if $a \in C$ and $p \in P$, then $a\bar{\varrho}(p)o$, $o\bar{\varrho}(p)a$, $o\bar{\varrho}(p)o$, $o\bar{\varrho}(p)I$, $I\varrho(p)o$, $I\varrho(p)I$ and if $a \in C(p)$, then $a\varrho(p)I$, $I\varrho(p)a$.

We can see that if $a \in C(p)$, $(p \in P)$ and $b \in C$, then $(a \vee b)\,\varrho(p)a$, $(I \vee \vee\, b)\varrho(p)I$, $(a \vee I)\varrho(p)a$, $(I \vee b)\varrho(p)I$, $(I \vee I)\varrho(p)I$. Hence if $x \in C(p) \cup I$ and $y \in C \cup I$, then $y \vee x \in C(p) \cup I$. Therefore $\big(C(p) \cup I, V\big)$ is an ideal of $(C \cup I, V\ I, V)$ and

$$(C \cup I, V) = \bigcup_{p \in P} (C(p) \cup I, V). \tag{38}$$

If $a\varrho(p)b$, then $(a \vee c)\varrho(p)(b \vee c)$, $(a, b, c \in C)$, but for arbitrary a, b elements, of C

$$(a \vee I)\,\varrho(p)(b \vee I). \tag{39}$$

Now we suppose that for all a elements of C, $a < I$, $o < a$, $o < I$. If $x, y, z \in C \cup o \cup I$ and $x \leqq y$, then $x \vee z \leqq y \vee z$, but $x \wedge z \leqq y \wedge z$ generally does not hold. If $a \in C(p)$ and $a \leqq b$ then $b \in C(p)$, from which $a\varrho(p)b$ follows.

We shall say that m is a maximal element of C if $a \geqq m$; $(a \in C)$ holds if $a = m$. Denote by M the set of all maximal elements of C.

If $a \in C$, then denote by $R(a)$ the set of all x elements of C for which $x \leqq a$. If $x, y \in R(a)$, then

$$x \vee y = \begin{cases} x \cup y & \text{if} \quad x \cup y \in C, \\ I & \text{if} \quad x \cup y \notin C, \end{cases} \tag{40}$$

and

$$x \wedge y = \begin{cases} x \cap y & \text{if} \quad x \cap y \in C, \\ o & \text{if} \quad x \cap y \notin C, \end{cases} \tag{41}$$

further

$$o \wedge I = I \wedge o = o,\ I \wedge I = I,\ x \wedge o = o \wedge x = o,\ o \wedge o = o, \tag{42}$$

$$I \wedge x = x \wedge I = x,\ o \vee I = I \vee o = I,\ I \vee I = I,\ x \vee o = o \vee x = x,$$

$$o \vee o = o,\ I \vee x = x \vee I = I.$$

If $x, y \in R(a) \cup o \cup I$, then

$$x \wedge y = y \wedge x \quad (43) \qquad \text{and} \qquad x \vee y = y \vee x, \tag{44}$$

further

$$(x \wedge y) \vee x = x \quad (45) \qquad \text{and} \qquad (x \vee y) \wedge x = x. \tag{46}$$

If $x, y, z \in R(a) \cup o \cup I$, then generally

$$(x \wedge y) \wedge z \neq x \wedge (y \wedge z) \quad (47) \quad \text{and} \quad (x \vee y) \vee z \neq x \vee (y \vee z). \tag{48}$$

Therefore we obtain the following theorem:

Theorem 3. If M is the set of all maximal elements of C, then

$$(C \cup o \cup I, \wedge, \vee) = \bigcup_{m \in M} (R(m) \cup o \cup I, \wedge, \vee). \tag{49}$$

Remarks. If the conditions 1°, 2°, do not hold, then $(C \cup o \cup I, \wedge, \vee)$ is not a lattice, but the decomposition

holds, where generally $(R(m) \cup o \cup I, \wedge, \vee)$ are not lattices. (49)

We can see that the following relations do not depend from the conditions 1°, 2°:

If $x, y, z \in C \cup o \cup I$, then $x \wedge y$, $x \vee y \in C \cup o \cup I$, and

$$x \wedge y = y \wedge x, \quad (50) \qquad x \vee y = y \vee x, \quad (51)$$

$$x \wedge (x \vee y) = x, \quad (52) \qquad x \vee (x \wedge y) = x. \quad (53)$$

On the structure $(\mathbf{P} \cup \mathbf{i}, \sqcap, \sqcup)$

If $p, q \in P$, then

$$p \sqcup q = q \sqcup p \tag{54}$$

and

$$p = p \sqcup i = i \sqcup p, \; i \sqcup i = i. \tag{55}$$

If $x, y, z \in P \cup i$ and $i \in \{x, y, z\}$, then

$$(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z). \tag{56}$$

If $p, q \in P$, then

$$p \sqcap q = q \sqcap p \tag{57}$$

and

$$p \sqcap i = i \sqcap p = i \sqcap i = i, \tag{58}$$

further

$$(p \sqcap q) \sqcup p = \begin{cases} (p \cap q) \cup p = p & \text{if} \quad p \cap q \in P, \\ i \sqcup p = p & \text{if} \quad p \cap q \notin P, \end{cases} \tag{59}$$

and

$$(p \sqcap i) \sqcup i = i, \; (i \sqcap i) \sqcup i = i,$$
$$(i \sqcap p) \sqcup p = p, \; i \sqcup p = p. \tag{60}$$

If $p, q, \in P$, then

$$(p \sqcup p) \sqcap p = (p \cup q) \cap p = p, \tag{61}$$

and

$$(p \sqcup i) \sqcap i = i, \; p \sqcap i = i, \; p \sqcap p = p,$$
$$(p \sqcup i) \sqcap p = p, \; (i \sqcup i) \sqcap i = i. \tag{62}$$

If $p, q, r \in P$, then

$$(p \sqcup q) \sqcup r = p \sqcup (q \sqcup r) = p \cup q \cup r, \tag{63}$$

and if $x, y, z \in P \cup i$, then

$$(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z). \tag{64}$$

If $p, q, r \in P$, then it can be proved that

$$(p \sqcap q) \sqcap r = p \sqcap (q \sqcap r) \tag{65}$$

if and only if the following conditions hold:

3° $p \cap q \cap r \notin P$ or if $p \cap q \cap r \in P$, then either both $p \cap q$ and $q \cap r$ belong to P, or none of them belongs to P.

If $x, y, z \in P \cup i$, then

$$(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z). \tag{66}$$

Therefore we have the following theorem:

Theorem 4. The elements of the set $P \cup i$ under the operations $\sqcap$, $\sqcup$ form a lattice if and only if the condition 3° holds.

Denote by $P(a)$; $(a \in C)$ the set of all $p \in P$ problems which can be solved by a. If $a, b, b \cap c \in C$, then $P(a \cap b) \subseteq P(a) \cap P(b)$ and *if* $a, b, a \cup \cup b \in C$, then $P(a \cup b) \supseteq P(a) \cup P(b)$.

We can see that if $a \in C$ and $p, q \in P$, then

$$p\sigma(a)q \Leftrightarrow p, q \in P(a), \tag{67}$$

and if $a, b \in C$, and $p, r \in P(a)$ further $q, r \in P(b)$, then $p(\sigma(a)\sigma(b))q$. If $a, b \in C$ and $p, q \in P(a)$ or $p, q \in P(b)$, then $p(\sigma(a) + \sigma(b))q$. If $p \in P(a)$; $(a \in C)$ and $q \in P$, then

$$p \sqcap q = \begin{cases} p \cap q & \text{if} \quad p \cap q \in P, \\ i & \text{if} \quad p \cap q \notin P, \end{cases} \tag{68}$$

and $q \sqcap i = i \sqcap q = i \sqcap i = i$. Therefore $(P(a) \cup i, \sqcap)$ is an ideal of the groupoid $(P \cup i, \sqcap)$ and

$$(P \cup i, \sqcap) = \bigcup_{a \in C} (P(a) \cup i, \sqcap). \tag{69}$$

Denote by $\widehat{P}$ the set of all $P(a)$; $(a \in C)$ sets.

On the $P \cup \widehat{P} \cup i$ set we define two operations as follows: if $a, b \in C$, then

$$P(a) \wedge P(b) = \begin{cases} P(a \cap b) & \text{if} \quad a \cap b \in C, \\ i & \text{if} \quad a \cap b \notin C, \end{cases} \tag{70}$$

$$P(a) \wedge i = i \wedge P(a) = i,\ i \wedge i = i,\ P \wedge P(a) = P(a) \wedge P = \tag{71}$$

$$= P(a),\ P \wedge i = i \wedge P = i,\ P \wedge P = P,$$

$$P(a) \vee P(b) = \begin{cases} P(a \cup b) & \text{if} \quad a \cup b \in C, \\ i & \text{if} \quad a \cup b \notin C, \end{cases} \tag{72}$$

(73) $$P(a) \vee i = i \vee P(a) = P(a),\ i \vee i = i,\ P \vee P(a) =$$

$$= P(a) \vee P = P,\ P \vee i = i \vee P = P,\ P \vee P = P.$$

If $a, b \in C$, then

$$\big(P(a) \wedge P(b)\big) \vee P(a) = \begin{cases} P(a \cap b) \vee P(a) = P(a) & \text{if} \quad a \cap b \in C, \\ i \vee P(a) = P(a) & \text{if} \quad a \cap b \notin C \end{cases} = P(a), \tag{74}$$

and

$$\big(P(a) \vee\big) P(b) \wedge P(a) = \begin{cases} P(a \cup b) \wedge P(a) = P(a) & \text{if} \quad a \cup b \in C, \\ P \wedge P(a) = P(a) & \text{if} \quad a \cup b \notin C \end{cases} = P(a), \tag{75}$$

further

$$(P(a) \wedge i) \wedge i = (P(a) \vee i) \wedge i = i,$$
$$(P(a) \wedge P) \vee P = (P(a) \vee P) \wedge P = P, \tag{76}$$
$$(P \wedge i) \vee i = (P \vee i) \wedge i = i.$$

Therefore if $x, y, \in \widehat{P} \cup P \cup i$, then

(77) $(x \wedge y) \vee x = x$ and (78) $(x \vee y) \wedge x = x.$

It can be proved — as we did previously — that $(\widehat{P} \cup P \cup i, \wedge, \vee)$ is a lattice if and only if for any triplet $a, b, c \in C$, the following two conditions hold:

4° $a \cap b \cap c \notin C$ or if $a \cap o \cap c \in C$, then either both $a \cap b$ and $b \cap c$ belong to C or none of them belongs to C,

5° $a \cup b \cup c \notin C$ or if $a \cup b \cup c \in C$, then either both $a \cup b$, $b \cup c$ belong to C or none of them belongs to C.

Entropy of the pairs (p, C) and (a, P)

We can see that the following entropy concepts are very useful tools at the analysis of the structures $(C \cup o \cup I, \wedge, \vee)$, $(P \cup i, \sqcap, \sqcup)$.

If H is a finite set, then denote by $\mu(H)$ the number of the elements of H.

We suppose that C and P are finite sets.

If $p \in P$ then we define the entropy of the pair (p, C) as follows:

$$\xi(p, C) = -\frac{\mu(C(p))}{\mu(C)} \smile^{2} \log \frac{\mu(C(p))}{\mu(C)} - \\ -\frac{\mu(C) - \mu(C(p))}{\mu(C)} \smile^{2} \log \frac{\mu(C) - \mu(C(p))}{\mu(C)}. \tag{79}$$

If $a \in C$, then we define the entropy of the pairs (a, P) as follows:

$$\xi(a, P) = -\frac{\mu(P(a))}{\mu(P)} \smile^{2} \log \frac{\mu(P(a))}{\mu(P)} - \\ -\frac{\mu(P) - \mu(P(a))}{\mu(P)} \smile^{2} \log \frac{\mu(P) - \mu(P(a))}{\mu(P)}. \tag{80}$$

Finally we set up two open problems.

Problem 1. Determine the basic properties of $\xi(p, C)$ when $(C \cup o \cup I, \wedge, \vee)$ is a lattice.

Problem 2. Determine the basic properties of $\xi(a, P)$ when $(P \cup i, \sqcap, \sqcup)$ is a lattice.

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