ON THE MONOTONE SOLUTIONS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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There are many results known about the oscillation of the solution of the equation

$$y'' + a(t)f(y) = 0 \tag{1}$$

Let us consider the conditions in paper [1]:
Let $f(u)$ be a continuous, differentiable function on $(-\infty, 0) \cup (0, \infty)$, $uf(u) > 0$, $u \neq 0$, and $f'(u) \geq 0$, furthermore for $x > 0$ let

$$\int_x^\infty \frac{du}{f(u)} < \infty, \quad \int_{-x}^{-\infty} \frac{du}{f(u)} < \infty. \tag{2}$$

In the case of the fulfilment of the condition

$$\int_x^\infty ta(t)\mathrm{d}t = \infty \tag{3}$$

each solution y is either oscillating or tends to zero monotonously.

Consider now the equation

$$y'' + a(t)\, f\big(y(\tau(t))\big) = 0 \tag{4}$$

where $\tau(t) = t - \Delta t$, $\Delta t \geq 0$ and $\Delta t \in C[t_0, \infty)$,

$$\lim_{t\to\infty} \tau(t) = +\infty \tag{5}$$

$$\tau'(t) \geq \alpha > 0 \tag{6}$$

for $t \in [t_0, \infty)$.

Theorem 1. Let

a) $a(t) \in C[t_0, \infty)$;

b) $f(u) \in C^1(-\infty, \infty)$, $uf(u) > 0$, $u \neq 0$, $f'(u) \geqslant 0$;

c) For each $x > 0$ condition (2) is satisfied $\displaystyle\int_{-x}^{x} \frac{du}{f(u)}$

Suppose that

$$\int_x^\infty \tau(t)\, a(t)\, dt = +\infty \tag{7}$$

then the equation has no monotone solution.

Proof. Suppose that $y(t)$ is a monotone solution and $y(t) > 0$ on the interval $[t_0, \infty)$, then according to (5) $y(\tau(t)) > 0$ for some $t \geq t_1$. Multiplying both sides of equation (4) with $\tau(t)/f(y(\tau(t)))$ and integrating partially from t_1 to t we obtain:

$$\frac{\tau(s)y'(s)}{f(y(\tau(s)))}\Bigg|_{t_1}^{t} - \int_{t_1}^{t} \frac{y'(s)\tau'(s)}{f(y(\tau(s)))}\, ds + \int_{t_1}^{t} \frac{f'(y(\tau))y'(\tau)\tau'(s)\tau(s)y'(s)}{f^2(y(\tau(s)))}\, ds = -\int_{t_1}^{t} \tau(s)a(s)ds$$

$$\frac{\tau(t)y'(t)}{f(y(\tau(t)))} = \int_{t_1}^{t} \frac{y'(s)\tau'(s)}{f(y(\tau(s)))}\, ds - \int_{t_1}^{t} \frac{f'(y(\tau))y'(\tau)\tau'(s)\tau(s)y'(s)}{f^2(y(\tau(s)))}\, ds -$$

$$-\int_{t_1}^{t} \tau(s)a(s)ds + \frac{\tau(t_1)y'(t_1)}{f(y(\tau(t_1)))},$$

$$\frac{\tau(t)y'(t)}{f(y(\tau(t)))} \leq \int_{t_1}^{t} \frac{y'(s)\tau'(s)}{f(y(\tau(s)))}\, ds - \int_{t_1}^{t} \tau(s)a(s)ds + \frac{\tau(t_1)y'(t_1)}{f(y(\tau(t_1)))}$$

$$\frac{\tau(t)y'(\tau(t))}{f(y(\tau(t)))} \leq \int_{t_1}^{t} \frac{y'(\tau(s))\tau'(s)}{f(y(\tau(s)))}\, ds - \int_{t_1}^{t} \tau(s)a(s)ds + \frac{\tau(t_1)y'(t_1)}{f(y(\tau(t_1)))}$$

$$\frac{\tau(t)y'(\tau(t))}{f(y(\tau(t)))} \leq \int_{y(\tau(t_1))}^{y(\tau(t))} \frac{ds}{f(s)} - \int_{t_1}^{t} \tau(s)a(s)ds + \frac{\tau(t_1)y'(t_1)}{f(y(\tau(t_1)))}. \tag{8}$$

The first integral on the right side is bounded from above because of (2) and on the ground of (7) the inequality (8) implies

$$\frac{\tau(t)y'(\tau(t))}{f(y(\tau(t)))} \to -\infty,$$

when $t \to \infty$.
Thus for a $k > 0$ and $t_2 \geqslant t_1$

$$\frac{\tau(t)y'(\tau(t))}{f(y(\tau(t)))} < -k \tag{9}$$

respectively

$$\frac{\tau(t)y'(\tau(t))\tau'(t)}{f(y(\tau(t)))} < -k\tau'(t)$$

i.e.

$$\frac{y'(\tau(t))\tau'(t)}{f(y(\tau(t)))} < -k\frac{\tau'(t)}{\tau(t)}$$

for every $t \geq t_2 \geq t_1$. Integrating from t_2 to t we obtain

$$\int\limits_{y(\tau(t_2))}^{y(\tau(t))} \frac{ds}{f(s)} < \ln\left[\frac{\tau(t_2)}{\tau(t)}\right]^k. \tag{10}$$

The expression on the right side tends to $-\infty$ when $t \to \infty$. According to the properties of $y(t)$ and the assumptions we have a contradiction. Thus the theorem is proved.

I. V. Kamenyev investigated the oscillation properties of the nonlinear second-order differential equation

$$[r(x)y']' + a(x)'f(y) = 0 \tag{11}$$

under the conditions $f(u)$ is a continuously differentiable function on the interval $(-\infty, 0) \cup (0, \infty)$ $uf(u) > 0$, $u \neq 0$, $f'(u) \geq 0$ and for each $\varepsilon > 0$

$$\int\limits_{\varepsilon}^{\infty}\frac{du}{f(u)} < \infty\,, \quad \int\limits_{-\varepsilon}^{-\infty}\frac{du}{f(u)} < \infty\,, \qquad \varepsilon > 0 \quad (F) \tag{12}$$

$$\int\limits_{0}^{\varepsilon}\frac{du}{f(u)} < \infty\,, \quad \int\limits_{0}^{-\varepsilon}\frac{du}{f(u)} < \infty\,, \qquad \varepsilon > 0 \quad (\Phi) \tag{13}$$

as well as $0 < \varrho(x) \in C^2[x_0, \infty)$ is such a function that

$$\int^{+\infty} \varrho(x)\, a(x)\, dx = \infty \tag{14}$$

$$\int^{\infty}\frac{dx}{\varrho(x)\, r(x)} = \infty \tag{15}$$

furthermore

$$\varrho'(x) \geq 0; \quad R'(x) \leq 0 \qquad (F) \tag{16}$$

$$\left.\begin{aligned} &\varrho'(x) \leq 0; \quad R'(x) \geq 0 \\ &\int^{\infty} |R'(x)|dx < \infty \end{aligned}\right\} \quad (\Phi) \tag{17, 18}$$

where $R(x) = r(x)\varrho'(x)$.

Consider now the equation

$$[r(t)y']' + a(t)f\big(y(\tau(t))\big) = 0 \tag{19}$$

Theorem 2. Let

a) $0 < r(t) \in C^1[t_0, \infty)$;

b) $a(t) \in C[t_0, \infty)$;

c) $f(u) \in C^1[(-\infty, 0) \cup (0, \infty)]$, $uf(u) > 0$, $u \neq 0$ and $f'(u) \geq 0$;

d) $0 < \varrho(t) \in C^2[t_0, \infty)$ and such a function that

$$\int^{\infty} \varrho\big(\tau(t)\big)\, a(t)\, dt = \infty \tag{20}$$

$$\int^{\infty} \frac{dt}{\varrho\big(\tau(t)\big)\, r(t)} = \infty\,. \tag{21}$$

Furthermore (5), (6), (12) and (13) are satisfied as well as

$$\varrho'\big(\tau(t)\big) \geq 0; \;\; R'(t) \leq 0 \tag{22}$$

or

$$\varrho'\big(\tau(t)\big) \leq 0; \;\; R'(t) \geq 0 \tag{23}$$

or

$$\int^{\infty} |R'(u)| du < \infty \tag{24}$$

where $R(t) = r(t)\varrho'(\tau(t))\tau'(t)$ which are assuring separate cases in which the equation (19) has no monotonous solution.

Proof. Before we would turn to the proof we make a preliminary note. In the case when condition (13) is satisfied, for every u

$$F(u) = \int_0^u \frac{d(t)}{f(t)}$$

and $F(u) \geq 0$.

Now we can turn to the proof, suppose that equation (19) possesses monotone solution $y(t)$ under the following conditions

$$y(t) > 0 \text{ for every } t \geq t_1 > t_0. \tag{25}$$

From (19) multiplying it with $\varrho\big(\tau(t)\big)/f\big(y(\tau(t))\big)$ and integrating from t_1 to t we obtain:

$$\frac{r(s)\varrho\big(\tau(s)\big)y'(s)}{f\big(y(\tau(s))\big)}\Bigg|_{t_1}^{t} - \int_{t_1}^{t} \frac{r'(s)\varrho\big(\tau(s)\big)y'(s)}{f\big(y(\tau(s))\big)}\, ds - \int_{t_2}^{t} \frac{r(s)\varrho'\big(\tau(s)\big)\tau'(s)y'(s)}{f\big(y(\tau(s))\big)}\, ds +$$

$$+ \int_{t_1}^{t} \frac{r(s)\varrho\big(\tau(s)\big)f'\big(y(\tau)\big)y'(\tau)\tau'(s)y'(s)}{f^2\big(y(\tau(s))\big)}\, ds + \int_{t_1}^{t} \frac{r'(s)\varrho\big(\tau(s)\big)y'(s)}{f\big(y(\tau(s))\big)}\, ds = -\int_{t_1}^{t} \varrho\big(\tau(s)\big)a(s) ds,$$

respectively

$$\frac{r(t)\varrho(\tau(t))y'(t)}{f(y(\tau(t)))}+\int_{t_1}^{t}\frac{r(s)\varrho(\tau(s))y'(s)f'(y(\tau(s)))y'(\tau(s))\tau'(s)}{f^2(y(\tau(s)))}ds=$$

$$=\int_{t_1}^{t}\frac{r(s)\varrho'(\tau(s))\tau'(s)y'(s)}{f(y(\tau(s)))}ds+C-\int_{t_1}^{t}\varrho(\tau(s))a(s)ds. \tag{26}$$

Since $f'(y(\tau))\geq 0$ therefore

$$\frac{r(t)\varrho(\tau(t))y'(\tau(t))}{f(y(\tau(t)))}\leq C+W(t)-\int_{t_1}^{t}\varrho(\tau(s))a(s)ds \tag{27}$$

where

$$W(t)=\int_{t_1}^{t}r(s)\varrho'(\tau(s))\tau'(s)\frac{y'(\tau(s))}{f(y(\tau(s)))}ds=\int_{t_1}^{t}R(s)\frac{y'(\tau(s))}{f(y(\tau(s)))}ds.$$

Now we prove that conditions (22)—(24) assure the boundedness of $W(t)$. Therefore let us consider the following three cases:

1. Let condition (22) be fulfilled then on the ground of Bohne's theorem

$$W(t)=\frac{R(t_1)}{\alpha}\int_{t_1}^{t}\frac{y'(\tau(s))\tau'(s)}{f(y(\tau(s)))}ds=\frac{R(t_1)}{\alpha}\int_{y(\tau(t_1))}^{y(\tau(\xi))}\frac{dz}{f(z)}\leq C_1.$$

2. Let condition (23) be satisfied, then on the ground of Bohne's theorem

$$W(t)=\frac{R(t_1)}{\alpha}\int_{t_1}^{\xi}\frac{y'(\tau(s))\tau'(s)}{f(y(\tau(s)))}ds=\frac{R(t_1)}{\alpha}\int_{y(\tau(t_1))}^{y(\tau(\xi))}\frac{dz}{f(z)}\leq C_2.$$

3. Let condition (24) be satisfied

$$W(t)=\frac{1}{\alpha}\int_{t_1}^{t}R(s)\frac{y'(\tau(s))\tau'(s)}{f(y(\tau(s)))}ds=C_3'+\frac{1}{\alpha}R(t)F(y(\tau(t)))-$$

$$-\frac{1}{\alpha}\int_{t_1}^{t}F(y(\tau(s)))R'(s)ds\leq C_4=C_3+MN+M\int_{t_1}^{t}|R'(s)|ds\,,\left(C_3=\frac{C_3'}{\alpha}\right),$$

where $0\leq F(y(\tau(t)))\leq M$, $|R(t)|\leq N$ respectively $R(t)F(y(\tau))\leq MN$. Then there is such a D that $W(t)\leq D$ and thus from (27) we obtain

$$\lim_{t\to\infty}\frac{r(t)\varrho(\tau(t))y'(\tau(t))}{f(y(\tau(t)))}=-\infty$$

respectively

$$y'(\tau(t)) < 0. \tag{28}$$

Thus $y'(t) \leq 0,\ t \geq t_2 \geq t_1$.
Let $t_3 \geq t_2$, such that

$$C + D - \int_{t_1}^{t} \varrho(\tau(s))a(s)ds \leq -k, \quad t \geq t_3,$$

then (26) implies

$$\frac{r(t)\varrho(\tau(t))y'(t)}{f(y(\tau(t)))} + \int_{t_2}^{t} \frac{r(s)\varrho(\tau(s))y'(s)f'(y(\tau(s)))y'(\tau(s))\tau'(s)}{f^2(y(\tau(s)))}\,ds \leq -k,$$

or

$$K + \int_{t_3}^{t} \frac{r(s)\varrho(\tau(s))y'^2(\tau(s))f'(y(\tau(s)))\tau'(s)}{f^2(y(\tau(s)))}\,ds \leq \frac{r(t)\varrho(\tau(t))[-y'(\tau(t))]}{f(y(\tau(t)))}, \quad t \geq t_3. \tag{29}$$

Extending and integrating (29) we arrive at

$$\frac{r(t)\varrho(\tau)[-y'(\tau)]}{f(y(\tau))} \cdot \frac{f'(y(\tau))[-y'(\tau)]\tau'}{f(y(\tau))} \geq$$

$$\geq \frac{f'(y(\tau))[-y'(\tau)]\tau'}{f(y(\tau))} \cdot \left(K + \int_{t_3}^{t} \frac{r(s)\varrho(\tau)[-y'(\tau)]}{f(y(\tau))} \cdot \frac{f'(y(\tau))[-y'(\tau)]\tau'}{f(y(\tau))}\,ds\right)$$

respectively

$$\frac{\dfrac{r(t)\varrho(\tau)[-y'(\tau)]}{f(y(\tau))} \cdot \dfrac{f'(y(\tau))[-y'(\tau)]\tau'}{f(y(\tau))}}{K + \displaystyle\int_{t_3}^{t} \frac{r(s)\varrho(\tau)[-y'(\tau)]}{f(y(\tau))} \cdot \frac{f'(y(\tau))[-y'(\tau)]\tau'}{f(y(\tau))}\,ds} \geq \frac{f'(y(\tau))[-y'(\tau)]\tau'}{f(y(\tau))}$$

$$\ln\left(K + \int_{t_3}^{t} \frac{r(s)\varrho(\tau)[-y'(\tau)]}{f^2(y(\tau))} f'(y(\tau))\tau'ds\right) \geq \ln \frac{f(y(\tau(t_3)))}{f(y(\tau(t)))}, \quad t \geq t_3$$

i.e.

$$\frac{f(y(\tau(t_3)))}{f(y(\tau(t)))} \leq K + \int_{t_3}^{t} \frac{r(s)\varrho(\tau)y'^2(\tau)}{f^2(y(\tau))} f'(y(\tau))\tau'ds.$$

Making use of (29) we obtain

$$\frac{f(y(\tau(t_3)))}{f(y(\tau(t)))} \leq \frac{r(t)\varrho(\tau(t))[-y'(\tau(t))]}{f(y(\tau(t)))}, \quad t \geq t_3, \tag{30}$$

respectively

$$y'(t) \leq y'(\tau(t)) \leq -f(y(\tau(t_3)))\frac{1}{\varrho(\tau(t))r(t)}.$$

From (30) making use of condition (21)

$$y(t) \leq y(t_3) - f(y(\tau(t_3)))\int_{t_3}^{t}\frac{ds}{\varrho(\tau(s))r(s)} \to -\infty$$

arises, when $t \to \infty$, which is in contradiction with (25). Thus the theorem is proved. The case, when $y(t) < 0$, is proved in the very similar way.

Literature

1. ONOSE, H.: Oscillation theorems for nonlinear second order differential equations. Proc. Amer. Math. Soc. *26* (1970), 461—464.
2. КАМЕНЬЕВ, И. В.: Дифференциальные уравнения, № 9 (1970), 1718—1721.

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