CRITERIA FOR MONOTONE SOLUTIONS OF SOME SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Received Aug. 9, 1988

In this work we give an introductory short survey over an oscillatory theorem of A. Wintner and P. Hartman concerning second order linear differential equations, as well as over the Kamenyev generalization of the mentioned theorem and apply it to second order nonlinear delayed argument differential equations and their solutions.

There are many results known in connection with the oscillation of the solution of the equation

$$y'' + a(x)y = 0. \tag{1}$$

A. Wintner proved in [1] that in the case of fulfilment of condition

$$\lim_{x\to+\infty} A(x) = +\infty \tag{2}$$

where

$$A(x) = \frac{1}{x}\int_{x_0}^{x} dt \int_{x_0}^{t} a(s)ds \tag{3}$$

every solution of equation (1) is oscillatory.

P. Hartman showed in [2] that condition (2) may be replaced by condition

$$\lim_{x\to+\infty} \sup A(x) = +\infty \tag{4}$$

where $A(x)$ is the same as in (3).

I. V. Kamenyev in his works [7], [8] transformed both of these criteria into a more general form for the nonlinear equation

$$y'' + a(x)f(y) = 0 \tag{5}$$

establishing clearly understandable stipulations concerning the functions occurring in the equation.

Let us consider the conditions of [7], let $a(x) \in C[x_0, \infty)$, $f(y) \in C^1(-\infty, \infty)$, $\operatorname{sgn} f(y) = \operatorname{sgn} y$ while $f'(y) \geq \varepsilon > 0$ for $y \in R$, $0 < \psi(x) \in C[x_0, \infty)$ and satisfies the condition

$$\int^{\infty} \frac{\psi(x)\Theta(\int\limits_{x_0}^{x} \psi(s)ds)}{\int\limits_{x_0}^{x} \psi^2(s)ds} dx = +\infty \tag{6}$$

where $\Theta(x) \geq 0$ is continuous and nondecreasing for $x \geq 0$ and satisfies

$$\int^{+\infty} \frac{\Theta(x)}{x^2} dx < +\infty . \tag{7}$$

If the condition

$$\lim_{x \to +\infty} \frac{\int\limits_{x_0}^{x} \psi(t)\left(\int\limits_{x_0}^{t} a(s)ds\right)dt}{\int\limits_{x_0}^{x} \psi(s)ds} = +\infty \tag{8}$$

is satisfied then each solution of equation (5) is oscillatory.

Let us consider now the equation

$$y'' + a(t)f\big(y(\tau(t))\big) = 0 \tag{9}$$

where $\tau(t) = t - \Delta t$, $0 \leq \Delta t \in C[t_0, \infty)$

$$\lim_{t \to +\infty} \tau(t) = +\infty \tag{10}$$

$$\tau'(t) \geq \alpha > 0,\ t \in [t_0, \infty). \tag{11}$$

Theorem 1. Let

a) $a(t) \in C[t_0, \infty)$ and $a(t) \geq 0$ for $t \in [t_0, \infty)$;

b) $f(u) \in C^1(-\infty, \infty)$, $uf(u) > 0$, $f'(u) \geq 0$ for each u;

c) $0 < \psi(t) \in C[t_0, \infty)$ and satisfies the condition

$$\int^{+\infty} \frac{\psi\big(\tau(t)\big)\Theta\big(k \int\limits_{t_0}^{t} \psi(\tau(s))ds\big)}{\int\limits_{t_0}^{t} \psi^2\big(\tau(s)\big)ds} dt = +\infty \tag{12}$$

$\Theta(t) \geq 0$ is a nondecreasing continuous function for $t \geq 0$ and satisfies condition

$$\int^{+\infty} \frac{\Theta(t)}{t^2} dt < +\infty \tag{13}$$

Then, if the condition

$$\lim A_\varphi(t) = +\infty \tag{14}$$

is satisfied, where

$$A_\varphi(t) = \frac{\int\limits_{t_0} a(s) \int\limits_{x_0}^{s} \psi(\tau(u))du\, ds}{\int\limits_{t_0}^{t} \psi(\tau(s))ds} \tag{15}$$

hen equation (9) has no monotone solution.

Proof. Suppose that there is a monotone solution $y(t)$ of equation (9) and let $y(t) > 0$ for each $t \geq t_0$, then on the basis of (10) $y(\tau(t)) > 0$, $t \geq t_1$. Let us multiply the equation by the expression

$$\int\limits_{x_0}^{s} \psi(\tau(u))du/f(y(\tau(t)))$$

and let us integrate partially from t_0 to t, then

$$\frac{y'(s)}{f(y(\tau(s)))}\int\limits_{x_0}^{s}\psi(\tau(u))du\Big|_{t_0}^{t} - \int\limits_{t_0}^{t}\frac{y'(s)}{f(y(\tau(s)))}\psi(\tau(s))ds +$$

$$+\int\limits_{t_0}^{t}\frac{y'(s)f'(y(\tau(s)))y'(\tau(s))\tau'(s)}{f^2(y(\tau(s)))}\,ds\int\limits_{x_0}^{s}\psi(\tau(u))du = \int\limits_{t_0}^{t}a(s)\int\limits_{x_0}^{s}\psi(\tau(u))\,duds$$

i.e.

$$\int\limits_{x_0}^{s}\psi(\tau(u))du\int\limits_{t_0}^{t}\frac{y'(s)f'(y(\tau(s)))y'(\tau(s))\tau'(s)}{f^2(y(\tau(s)))}\,ds = \int\limits_{t_0}^{t}\frac{y'(s)}{f(y(\tau(s)))}\psi(\tau(s))ds -$$

$$-\frac{y'(t)}{f(y(\tau(t)))}\int\limits_{x_0}^{s}\psi(\tau(u))du + C\int\limits_{x_0}^{s}\psi(\tau(u))du - \int\limits_{t_0}^{t}a(s)\int\limits_{x_0}^{s}\psi(\tau(u))du\,ds$$

respectively

$$\frac{y'(t)}{f(y(\tau(t)))}\int\limits_{x_0}^{s}\psi(\tau(u))du + \int\limits_{x_0}^{s}\psi(\tau(u))du\int\limits_{t_0}^{t}\frac{y'(s)f'(y(\tau(s)))y'(\tau(s))\tau'(s)}{f^2(y(\tau(s)))}\,ds =$$

$$= \int\limits_{t_0}^{t}\frac{y'(s)}{f(y(\tau(s)))}\psi(\tau(s))ds + \left[C - \frac{\int\limits_{t_0}^{t}a(s)\int\limits_{x_0}^{s}\psi(\tau(u))du\,ds}{\int\limits_{t_0}^{t}\psi(\tau(s))ds}\right]\int\limits_{t_0}^{t}\psi(\tau(s))ds\,.$$

Using (15), from equation (9) we obtain

$$R(t)=\int_{t_0}^{t}\frac{y'(s)}{f(y(\tau(s)))}\psi(\tau(s))ds+[C-A_\psi(t)]\int_{t_0}^{t}\psi(\tau(s))ds \tag{16}$$

where

$$R(t)\equiv\int_{x_0}^{s}\psi(\tau(u))du\int_{t_0}^{t}z(s)ds,\quad z(t)=\frac{y'(\tau(t))y'(t)}{f^2(y(\tau(t)))}f'(y(\tau(t)))\tau'(t).$$

Since $y'(\tau(t))\geq y'(t)$ so $z(t)\geq z_1(t)$ where

$$z_1(t)=\left[\frac{y'(t)}{f(y(\tau(t)))}\right]^2 f'(y(\tau(t)))\tau'(t)\,.$$

Let $t_1>t_0$ such that $C-A_\psi(t)\leq 0$, if $t>t_1$ then from (16)

$$0\leq R(t)\leq\int_{t_0}^{t}\frac{y'(s)}{f(y(\tau(s)))}\psi(\tau(s))ds$$

is obtained for $t\geq t_1$ from which using the Bunyakovszkij inequality we can come to the conclusion

$$R^2(t)\leq\int_{t_0}^{t}z_1(s)ds\int_{t_0}^{t}\frac{\psi^2(\tau(s))}{f'(y(\tau))\tau'}ds\leq\frac{1}{\alpha\varepsilon}\int_{t_0}^{t}\psi^2(\tau(s))ds\int_{t_0}^{t}z_1(s)ds$$

i.e.

$$0<\frac{\alpha\varepsilon}{\int_{t_0}^{t}\psi^2(\tau(s))ds}\leq\frac{1}{R^2(t)}\int_{t_0}^{t}z_1(s)ds,\ t\geq t_1 \tag{17}$$

Since $\psi(\tau(t))\geq 0$, $z_1(t)\geq 0$ for $t\geq t_1$ we obtain

$$R(t)\geq\int_{x_0}^{s}\psi(\tau(u))du\int_{t_0}^{t}z_1(s)ds\geq k\int_{t_0}^{t}\psi(\tau(s))ds \tag{18}$$

where

$$k=\int_{t_0}^{t_1}z_1(s)ds>0.$$

From (18) we deduce

$$0\leq\Theta(k\int_{t_1}^{t}\psi(\tau(s))ds)\leq\Theta(R(t)),\quad t\geq t_1. \tag{19}$$

Multiplying (17) by (19) and by $\psi(\tau(t))$, and after that integrating it from t_1 to t with use of (13) we obtain

$$\frac{\beta\Theta\left(k\int\limits_{t_1}^{t}\psi(\tau(s))ds\right)}{\int\limits_{t_0}^{t}\psi^2(\tau(s))ds}\leq\frac{\Theta(R(t))}{R^2(t)},\quad(\beta=\alpha\varepsilon)$$

$$\frac{\beta\psi(\tau(t))\Theta\left(k\int\limits_{t_1}^{t}\psi(\tau(s))ds\right)}{\int\limits_{t_0}^{t}\psi^2(\tau(s))ds}\leq\frac{\Theta(R(t))}{R^2(t)}\cdot R'(t)$$

$$\beta\int\limits_{t_1}^{t}\frac{\psi(\tau(t))\Theta\left(k\int\limits_{t_1}^{t}\psi(\tau(s))ds\right)}{\int\limits_{t_0}^{t}\psi^2(\tau(s))ds}ds\leq\int\limits_{t_1}^{t}\frac{\Theta(R(s))}{R^2(s)}R'(s)ds=$$

$$=\int\limits_{R(t_1)}^{R(t)}\frac{\Theta(s)}{s^2}ds\leq\int\limits_{R(t_1)}^{\infty}\frac{\Theta(s)}{s^2}ds<\infty$$

which contradicts (12).

The case in which $y(t)<0$ is proved in a very similar way and thus the theorem is proved.

The conditions of [8] are globally equivalent to those of [7], the difference between them will play a role in the following theorem, but in a more extended form.

Theorem 2. Let the condition a) of Theorem 1. be satisfied and additionally let

b) $0<\varrho(t)\in C^1[t_0,\infty)$, $\varrho'(t)\leq 0$ and nondecreasing;

c) $0\leq\psi(\tau(t))\in C[t_0,\infty)$, $\psi(\tau(t))\not\equiv 0$, $\varrho(t)\psi(\tau(t))$ nondecreasing and

$$A_{\varrho,\psi}(t)=\frac{\int\limits_{t_0}^{t}a(s)\varrho(s)\int\limits_{x_0}^{s}\psi(\tau(u))du\,ds}{\int\limits_{t_0}^{t}\psi(\tau(s))ds};\tag{20}$$

d) $f(u)\in C^1(-\infty,\infty)$, $uf(u)>0$ and $f'(u)\geq 0$ for each u and

$$\int\limits_{\varepsilon}^{+\infty}\frac{du}{f(u)}<+\infty,\quad\int\limits_{-\varepsilon}^{-\infty}\frac{du}{f(u)}<+\infty,\qquad\varepsilon>0\tag{21}$$

$$\int\limits_{0}^{\varepsilon}\frac{du}{f(u)}<+\infty,\quad\int\limits_{0}^{-\varepsilon}\frac{du}{f(u)}<+\infty,\qquad\varepsilon>0.\tag{22}$$

If $\Phi(y) = \int\limits_0^y \frac{du}{f(u)} > 0$ for each y then the fulfilment of condition

$$\lim_{t\to\infty} \sup \ A_{\varrho,\psi}(t) = +\infty \tag{23}$$

implies that equation (9) has no monotone solution.

Proof. Suppose that equation (9) has a monotone solution $y(t)$, where $y(t) > 0$ for each $t \geq t_0$, then on the basis of (10) $y(\tau(t)) > 0$. We introduce the following substitution

$$y' = \frac{w(t)f(y(\tau(t)))}{\varrho(t) \int\limits_{t_0}^{k} \psi(\tau(s))ds} . \tag{24}$$

Differentiating (24) we obtain

$$y''(t) = \frac{w'f(y(\tau))}{\varrho \int\limits_{t_0}^{t} \psi(\tau)ds} + \frac{wf'(y(\tau))y'(\tau)\tau'}{\varrho \int\limits_{t_0}^{t} \psi(\tau(s))ds} - \frac{wf(y(\tau))\psi(\tau(t))}{\varrho[\int\limits_{t_0}^{t} \psi(\tau(s))ds]^2} - \frac{wf(y(\tau))\varrho'(t)}{\varrho^2(t) \int\limits_{t_0}^{t} \psi(\tau(s))ds} .$$

Then from equation (9) we obtain

$$-a(t)f(y(\tau)) = \frac{w'f(y(\tau))}{\varrho \int\limits_{t_0}^{t} \psi(\tau)ds} + \frac{wf'(y(\tau))y'(\tau)\tau'}{\varrho(t) \int\limits_{t_0}^{t} \psi(\tau(s))ds} - \frac{wf(y(\tau))\psi(\varepsilon(t))}{\varrho(t)[\int\limits_{t_0}^{t} \psi(\tau(s))ds]^2} - $$

$$- \frac{wf(y(\tau))\varrho'(t)}{\varrho^2(t) \int\limits_{t_0}^{t} \psi(\tau(s))ds} ,$$

after substituting $w(t)$ and multiplying by

$$\varrho(t) \int\limits_{t_0}^{t} \psi(\tau(s))ds/f(y(\tau))$$

and integrating from t_0 to t we arrive at

$$R(t) = w_1(t) + w_2(t) + [C - A_{\varrho,\psi}(t)] \int\limits_{t_0}^{t} \psi(\tau(s))ds \tag{25}$$

where

$$R(t) = \int\limits_{x_0}^{s} \psi(\tau(u))du \int\limits_{t_0}^{t} z(s)ds, \quad z(t) = \frac{y'(\tau(t))y'(t)}{f^2(y(\tau(t)))} \varrho(t)f'(y(\tau(t))) \geq 0$$

$$w_1(t) \equiv \int\limits_{t_0}^{t} \frac{\varrho(s)y'(s)}{f(y(\tau))} \psi(\tau(s))ds, \quad w_2(t) \equiv \int\limits_{t_0}^{t} \psi(\tau(u))du \int\limits_{t_0}^{t} \frac{y'(s)\varrho'(s)}{f(y(\tau(s)))} ds.$$

According to Bohne's mean-value theorem for $t \geq t_0$ and according to condition c)

$$w_1(t) = \int_{t_0}^{t} \frac{y'(s)}{f(y(\tau))}\, \varrho(s)\psi(\tau(s))ds \leq \frac{\varrho(t_0)\psi(\tau(t_0))}{\alpha} \int_{t_0}^{t} \frac{y'(\tau(s))\tau'(s)}{f\big(y(\tau(s))\big)}\, ds =$$

$$= \frac{\varrho(t_0)\psi(\tau(t_0))}{\alpha} \int_{y(\tau(t_0))}^{y(\tau(\xi))} \frac{du}{f(u)} = \frac{\varrho(t_0)\psi(\tau(t_0))}{\alpha}\, \Phi\, [y(\tau(t_0))] \leq d,$$

$$w_2(t) = \int_{t_0}^{t} \frac{y'(s)\varrho'(s)}{f\big(y(\tau(s)\big)}\, ds \leq \frac{\varrho'(t_0)}{\alpha} \int_{t}^{\xi} \frac{y'(\tau(s))\tau'(s)}{f\big(y(\tau(s))\big)}\, ds = \frac{\varrho'(t_0)}{\alpha} \int_{y(\tau(t_0))}^{y(\tau(\xi))} \frac{du}{f(u)} =$$

$$= \frac{\varrho'(t_0)}{\alpha}\, \Phi[y(\tau(t_0))] \leq D, \tag{26}$$

i.e.

$$w_2(t) = D \int_{t_0}^{s} \psi(\tau(u))du, \quad t \geq t_0. \tag{27}$$

If (26) and (27) are satisfied then from (25) we obtain

$$0 \leq R(t) \leq d + [C + D - A_{\varrho,\psi}(t)] \int_{t_0}^{t} \psi(\tau(s))ds \leq \infty, \quad t \geq t_0 \tag{28}$$

which contradicts condition (23). Thus we proved the theorem.

Theorem. 3. If the function $f(u)$ satisfies conditions (21) and (22) and the functions $\varrho'(t)$, $\varrho(t)\psi(\tau(t))$ are absolutely continuous while

$$\int_{0}^{\infty} |\varrho''(t)|dt < +\infty, \quad \int_{0}^{\infty} |[\varrho(t)\psi(\tau(t))]'|dt < +\infty \tag{29}$$

then condition (23) is sufficient to prove the fact that equation (9) has no monotonous solution.

Proof. Assume that there is a solution $y(t) > 0$ for $t \geq t_0$. Then similarly as in the proof of Theorem 2 we obtain equation (25). If (21), (22) and (29) are satisfied then for $t \geq t_0$.

$$w_1(t) = \int_{t_0}^{t} \frac{y'(s)}{f(y(\tau))}\, \varrho(s)\psi(\tau(s))ds \leq \frac{1}{\alpha} \int_{t_0}^{t} \frac{y'(\tau(s))\tau'(s)}{f\big(y(\tau(s))\big)}\, \varrho(s)\psi(\tau(s))ds =$$

$$= \frac{-c_1' + \varrho(t)\psi(\tau(t))\Phi\big(y(\tau(t))\big)}{\alpha} - \frac{1}{\alpha} \int_{t_0}^{t} \Phi(y(\tau))[\varrho(s)\psi(\tau(s))]'ds \leq$$

$$\leq d = |c_1| + c_2 \int_{t_0}^{\infty} |[\varrho(s)\psi(\tau(s))]'|ds \tag{30}$$

$$w_2(t) = \int\limits_{t_0}^{t} \frac{y'(s)\varrho'(s)}{f\big(y(\tau(s))\big)}\, ds \leq \frac{1}{\alpha} \int\limits_{t_0}^{t} \frac{y'(\tau(s))\tau'(s)}{f\big(y(\tau(s))\big)}\, \varrho'(s) ds =$$

$$= \frac{-c_3' + \varrho'(t)\Phi\big(y(\tau(t))\big)}{\alpha} - \frac{1}{\alpha} \int\limits_{t_0}^{t} \Phi\big(y(\tau(s))\big)\varrho''(s) ds \leq D =$$

$$= |c_3| + c_2 c_4 + c_2 \int\limits_{t_0}^{\infty} |\varrho''(s)| ds, \tag{31}$$

where

$$C_3 = \sup_{-\infty < y < +\infty} \Phi\big(y(\tau(s))\big), \qquad C_4 = \sup_{t_0 \leqq t < +\infty} |\varrho'(t_0)|$$

(31) implies (27), furthermore (27), (30) and (25) imply (28) which contradicts condition (23). Thus the theorem is proved.

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