THE STATICAL PROVING AND REFINEMENTS OF MARKOV'S INEQUALITY

L. SEBESTYÉN

Department of Mathematics. Faculty of Transport Engineering
Technical University H-1521, Budapest

Received Aug. 9, 1988

1. We are going to prove the following inequalities with statical methods.

a) If ξ is a non-negative random variable with the expected value M, then the following inequality holds (Markov):

$$\frac{M}{\varepsilon} \geqq P(\xi \geqq \varepsilon) = 1 - F(\varepsilon) \quad \text{if} \quad \varepsilon > 0. \tag{1}$$

b) If ξ is a non-negative random variable with the expected value M, and its distribution function $F(X)$ is concave in the section $(0, +\infty)$, then the following inequality holds:

$$\frac{M}{2\varepsilon} \geqq P(\xi \geqq \varepsilon) = 1 - F(\varepsilon), \quad \varepsilon > 0. \tag{2}$$

c) If ξ is a non-negative random variable with expected value M, and its distribution function $F(X)$ is concave in the section $(0, +\infty)$, then the following inequalities hold:

$$\frac{1}{\lambda^2} \geqq P(\xi \geqq \lambda M), \quad 0 < \lambda \leqq 2, \tag{3}$$

$$\frac{1}{\lambda^{2/3}} \geqq P(\xi \geqq \lambda M), \quad 0 < \lambda \leqq 4. \tag{4}$$

d) If ξ is a non-negative random variable with the expected value M, and variance δ^2, then the following inequalities hold:

$$\left(1 + \left(\frac{\sigma}{M}\right)^2\right) \frac{1}{\lambda^2} \geqq P(\xi \geqq 1 - F(\lambda M)), \ \lambda > 0, \tag{5}$$

$$\frac{1}{\lambda^2} \geqq P(\xi \geqq \lambda M), \quad \frac{\sigma}{M} \leqq 1, \qquad \lambda > 0, \tag{6}$$

In this lecture — for the sake of brevity — we shall only deal with inequalities (1), (2) and (3).

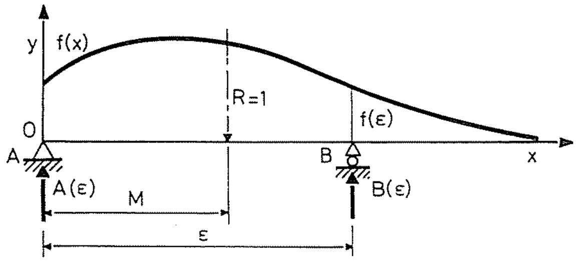

Fig. 1

2. For proving of inequality (1) let us consider a beam resting on supports at points A and B which has a span and an infinitely long cantilever at the right hand side. Let the function of its external load be denoted by $f(X)$ which is given non-negative function (Fig. 1). Let $A(\varepsilon)$ and $B(\varepsilon)$ denote the reactions acting at the respective supports.

Assuming that the load function is the density function of the non-negative random variable ξ, the resultant of the external load is equal to the unity, and its distance from point A is equal to the expected value of ξ, that is

$$\int_0^\infty f(x)dx = 1, \qquad \int_0^\infty xf(x)dx = M. \tag{7}$$

For expressing the reaction forces $A(\varepsilon)$ and $B(\varepsilon)$ we write a moment equilibrium equation for point A:

$$B(\varepsilon)\cdot\varepsilon - \int_0^\infty xf(x)dx = 0,$$

$$B(\varepsilon)\varepsilon = 1\cdot M,$$

hence

$$A(\varepsilon) = 1 - \frac{M}{\varepsilon}, \qquad B(\varepsilon) = \frac{M}{\varepsilon}. \tag{8}$$

Markov's inequality simply follows from the fact that the reaction force $B(\varepsilon) = \dfrac{M}{\varepsilon}$ cannot be less than the external load acting on the contilever that is,

$$\frac{M}{\varepsilon} \geqq \frac{1}{\varepsilon}\int_\varepsilon^\infty \varepsilon f(x)dx = 1 - F(\varepsilon) = P(\xi \geqq \varepsilon). \qquad \text{q.e.d.}$$

3. For proving inequality (3), let us consider a beam with two cantilevers, supported as seen in Fig. 2. Let the external load be the same as be-

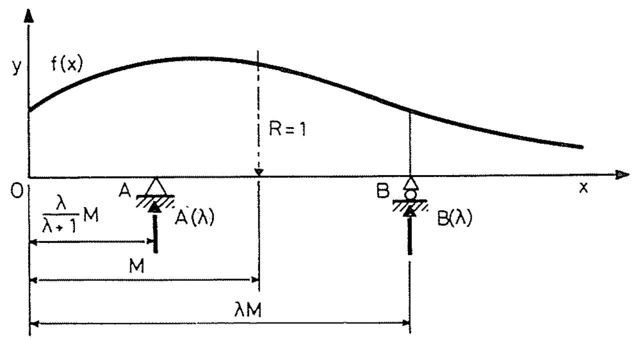

Fig. 2

fore expect that the origin of x is fixed at the end point of the cantilever at the left hand side.

$$B(\lambda)\left(\lambda M - \frac{\lambda}{\lambda+1}M\right) = 1 \cdot \left(M - \frac{\lambda}{\lambda+1}M\right).$$

Hence

$$B(\lambda)\left(\lambda - \frac{\lambda}{\lambda+1}\right) = 1 - \frac{\lambda}{\lambda+1}, \quad B(\lambda)\frac{\lambda^2}{\lambda+1} = \frac{1}{\lambda+1},$$

and

$$A(\lambda) = 1 - \frac{1}{\lambda^2}, \quad B(\lambda) = \frac{1}{\lambda^2}. \tag{9}$$

Now we must examine under what conditions reaction force $B(\lambda) = \frac{1}{\lambda^2}$ is not less than the resultant of the load acting on the cantilever at the right hand side. That is, under what conditions the inequality holds

$$B(\lambda) \geqq \int_{\lambda M}^{\infty} f(x)dx = 1 - F(\lambda M) = P(\xi \geqq \lambda M).$$

For that purpose let the external force of the beam be divided into three parts, and let the reaction forces be calculated separately for each part.

α) The external force acts on the section $\left(0, \frac{\lambda}{\lambda+1}M\right)$ of the beam (the hatched part of the diagram in Fig. 3).

The moment equilibrium equation for point A is:

$$\int^{\frac{\lambda}{\lambda+1}M} \left(\frac{\lambda}{\lambda+1}M - X\right) f(x)dx + B_\alpha(\lambda)\left(\lambda M - \frac{\lambda}{\lambda+1}M\right) = 0,$$

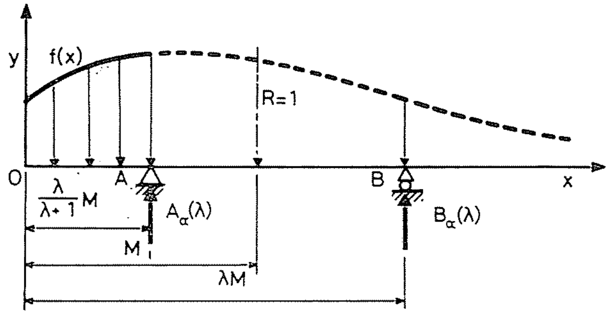

Fig. 3

$$\left[\left(\frac{\lambda}{\lambda+1}M - X\right)F(x)\right]_0^{\frac{\lambda}{\lambda+1}M} + \int\limits_0^{\frac{\lambda}{\lambda+1}M} F(x)dx + B_\alpha(\lambda)\frac{\lambda^2}{\lambda+1}M = 0,$$

$$B_\alpha(\lambda) = -\frac{\lambda+1}{\lambda^2 M}\int\limits_0^{\frac{\lambda}{\lambda+1}M} F(x)dx. \tag{10}$$

The equilibrium equation of forces is:

$$A_\alpha(\lambda) + B_\alpha(\lambda) - \int\limits_0^{\frac{\lambda}{\lambda+1}M} f(x)dx = 0,$$

$$A_\alpha(\lambda) = F\left(\frac{\lambda}{\lambda+1}M\right) + \frac{\lambda+1}{\lambda^2 M}\int\limits_0^{\frac{\lambda}{\lambda+1}M} F(x)dx. \tag{11}$$

3) The external force acts on the section $(\lambda M, +\infty)$ of the beam (the hatched part of the force diagram in Fig. 4).

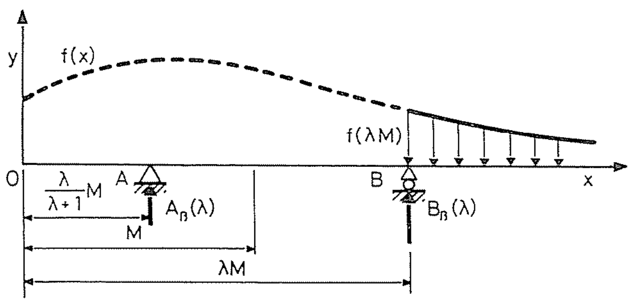

Fig. 4

The moment equilibrium equation for point A is:

$$B_\beta(\lambda)\left(\lambda M - \frac{\lambda}{\lambda+1}M\right) = \int_{\lambda M}^{\infty}\left(x - \frac{\lambda}{\lambda+1}M\right)f(x)dx,$$

$$B_\beta(\lambda)\frac{\lambda^2}{\lambda+1}M = \int_{\lambda M}^{\infty}(x-\lambda M)f(x)dx + \left(\lambda M - \frac{\lambda}{\lambda+1}M\right)(1 - F(\lambda M)),$$

$$B_\beta(\lambda)\frac{\lambda^2}{\lambda+1}M = \frac{\lambda^2}{\lambda+1}M(1 - F(\lambda M)) + \int_{\lambda M}^{\infty}(x-\lambda M)f(x)dx,$$

$$B_\beta(\lambda) = 1 - F(\lambda M) + \frac{\lambda+1}{\lambda^2 M}\int_{\lambda M}^{\infty}(x-\lambda M)f(x)dx. \tag{12}$$

The equilibrium equation of forces is:

$$A_\beta(\lambda) + B_\beta(\lambda) = \int_{\lambda M}^{\infty} f(x)dx = 1 - F(\lambda M)\,,$$

$$A_\beta(\lambda) = -\frac{\lambda+1}{\lambda^2 M}\int_{\lambda M}^{\infty}(x-\lambda M)f(x)dx. \tag{13}$$

The external force acts on the section $\left(\frac{\lambda}{\lambda+1}M, \lambda M\right)$ of the beam (the hatched part of the force diagram in Fig. 5.)

The moment equilibrium equation for point A is:

$$B_\gamma(\lambda)\left(\lambda M - \frac{\lambda}{\lambda+1}M\right) = \int_{\frac{\lambda}{\lambda+1}M}^{\lambda M}\left(x - \frac{\lambda}{\lambda+1}M\right)f(x)dx$$

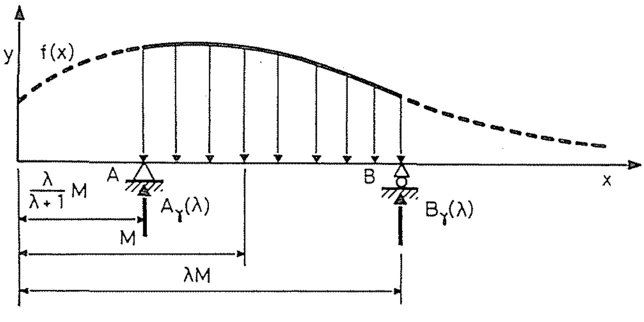

Fig. 5

$$B_\gamma(\lambda)\,\frac{\lambda^2}{\lambda+1}\,M=\left[\left(x-\frac{\lambda}{\lambda+1}\,M\right)F(x)\right]_{\frac{\lambda}{\lambda+1}M}^{\lambda M}-\int\limits_{\frac{\lambda}{\lambda+1}M}^{\lambda M}F(x)dx,$$

$$B_\gamma(\lambda)\,\frac{\lambda^2}{\lambda+1}\,M=\frac{\lambda^2}{\lambda+1}\,MF(\lambda M)-\int\limits_{\frac{\lambda}{\lambda+1}M}^{\lambda M}F(x)dx,$$

$$B_\gamma(\lambda)=F(\lambda M)-\frac{\lambda+1}{\lambda^2 M}\int\limits_{\frac{\lambda}{\lambda+1}M}^{\lambda M}F(x)dx. \tag{14}$$

The equilibrium equation of forces is:

$$A_\gamma(\lambda)+B_\gamma(\lambda)=F(\lambda M)-F\left(\frac{\lambda}{\lambda+1}\,M\right),$$

$$A_\gamma(\lambda)=\frac{\lambda+1}{\lambda^2 M}\int\limits_{\frac{\lambda}{\lambda+1}M}^{\lambda M}F(x)dx-F\left(\frac{\lambda}{\lambda+1}\,M\right). \tag{15}$$

We can check our results summing up the reactions as follows:

$$A_\alpha(\lambda)+A_\beta(\lambda)+A_\gamma(\lambda)=1-\frac{1}{\lambda^2},$$

$$B_\alpha(\lambda)+B_\beta(\lambda)+B_\gamma(\lambda)=\frac{1}{\lambda^2}.$$

By so doing we can write for reaction $B(\lambda)$ the following equation:

$$\frac{1}{\lambda^2}=-\frac{\lambda+1}{\lambda^2 M}\int\limits_{0}^{\frac{\lambda}{\lambda+1}M}F(x)dx+\big(1-F(\lambda M)\big)+\frac{\lambda+1}{\lambda^2 M}\int\limits_{\lambda M}^{\infty}(x-\lambda M)f(x)dx+$$

$$+F(\lambda M)-\frac{\lambda+1}{\lambda^2 M}\int\limits_{\frac{\lambda}{\lambda+1}M}^{\lambda M}F(x)dx,$$

$$\frac{1}{\lambda^2}=1-F(\lambda M)+F(\lambda M)+\frac{\lambda+1}{\lambda^2 M}\left(\int\limits_{\lambda M}^{\infty}(x-\lambda M)f(x)dx-\int\limits_{0}^{\lambda M}F(x)dx\right). \tag{16}$$

From this equation it follows that inequality $\frac{1}{\lambda^2} = 1 - F(\lambda M)$ holds if $\Phi_1(\lambda)$ is a non-negative value, where

$$\Phi_1(\lambda) = F(\lambda M) + \frac{\lambda + 1}{\lambda^2 M}\left(\int\limits_{\lambda M}^{\infty}(x - \lambda M)f(x)dx - \int\limits_{0}^{\lambda M} F(x)dx\right). \tag{17}$$

The sign of $\Phi_1(\lambda)$ is equal with that of $\Phi_2(\lambda)$, where

$$\Phi_2(\lambda) = \frac{\lambda^2}{\lambda + 1} MF(\lambda M) + \int\limits_{\lambda M}^{\infty}(x - \lambda M)f(x)dx - \int\limits_{0}^{\lambda M} F(x)dx = \frac{\lambda^2 M}{\lambda + 1}\Phi_1(\lambda). \tag{18}$$

With some rearrangements we can write:

$$\Phi_2(\lambda) = \frac{\lambda^2}{\lambda + 1} MF(\lambda M) + M - \lambda M(1 - F(\lambda M)) - \int\limits_{0}^{\lambda M} xf(x)dx - \int\limits_{0}^{\lambda M} F(x)dx,$$

$$\Phi_2(\lambda) = \frac{\lambda^2}{\lambda + 1} MF(\lambda M) - (\lambda - 1)M + \lambda MF(\lambda M) - \int\limits_{0}^{\lambda M} F(x)dx - \int\limits_{0}^{\lambda M} xf(x)dx,$$

$$\Phi_2(\lambda) = \frac{\lambda^2}{\lambda + 1} MF(\lambda M) - (\lambda - 1)M. \tag{19}$$

Now let us examine under what condition $\Phi_2(\lambda)$ will be non-negative.

If $\frac{1}{2\lambda} \geqq 1 - F(\lambda M)$, than

$$\Phi_2(\lambda) = \frac{\lambda^2}{\lambda + 1} MF(\lambda M) - (\lambda - 1)M \geqq \frac{\lambda^2}{\lambda + 1}\left(1 - \frac{1}{2\lambda}\right) M - (\lambda - 1)M,$$

$$\Phi_2(\lambda) \geqq \frac{M}{\lambda + 1}\left[\lambda^2\left(1 - \frac{1}{2\lambda}\right) - \lambda^2 + 1\right] = \frac{M}{\lambda + 1}\left[\lambda^2 - \frac{\lambda}{2} - \lambda^2 + 1\right],$$

$$\Phi_2(\lambda) \geqq \frac{M}{\lambda + 1}\left[1 - \frac{\lambda}{2}\right] \geqq 0, \quad \text{if} \quad 2 \geqq \lambda > 0. \quad \text{q.e.d.}$$

The proof of inequality (2) using more abstract mathematical considerations is as follows:

$$M = \int\limits_{0}^{\infty} xf(x)dx = \int\limits_{0}^{\varepsilon} xf(x)dx + \int\limits_{\varepsilon}^{\infty} xf(x)dx,$$

$$M = \int\limits_{0}^{\varepsilon} xf(x)dx + \int\limits_{\varepsilon}^{\infty} [\varepsilon + (x - \varepsilon)]f(x)dx,$$

$$M = \varepsilon(1 - F(\varepsilon)) + \int\limits_{0}^{\varepsilon} xf(x)dx + \int\limits_{\varepsilon}^{\infty} (x - \varepsilon)f(x)dx,$$

$$M = \varepsilon\big(1 - F(\varepsilon)\big) + \int_0^\varepsilon xf(x)dx + \int_\varepsilon^{2\varepsilon} (x - \varepsilon)f(x)dx + \int_{2\varepsilon}^\infty (x - \varepsilon)f(x)dx,$$

$$M \geqq \varepsilon\big(1 - F(\varepsilon)\big) + [XF(x)]_0^\varepsilon - \int_0^\varepsilon F(x)dx + [(x - \varepsilon)F(x)]_\varepsilon^{2\varepsilon} - \int_\varepsilon^{2\varepsilon} F(x)dx +$$

$$+ \varepsilon\big(1 - F(2\varepsilon)\big),$$

$$M \geqq \varepsilon\big(1 - F(\varepsilon)\big) + \varepsilon F(\varepsilon) + \varepsilon F(2\varepsilon) + - \varepsilon F(2\varepsilon) - \int_0^{2\varepsilon} F(x)dx,$$

$$M \geqq \varepsilon\big(1 - F(\varepsilon)\big) + \varepsilon - \varepsilon F(\varepsilon) + 2\varepsilon F(\varepsilon) - \int_0^{2\varepsilon} F(x)dx,$$

$$M \geqq 2\varepsilon\big(1 - F(\varepsilon)\big) + 2\varepsilon F(\varepsilon) - \int_0^{2\varepsilon} F(x)dx\,.$$

We can easily see in Fig. 6. that the inequality

$$2\varepsilon F(\varepsilon) - \int_0^{2\varepsilon} F(x)dx \geqq 0$$

holds, if the diagram of $F(x)$ is concave in section $(0, +\infty)$
In this way, the validity of inequation (2) is also proven.

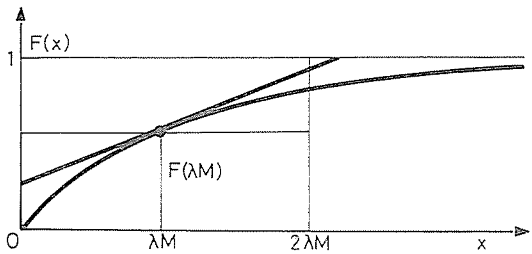

Fig. 6

The presented statical method can also be used for proving inequalities (4), (5) and (6).

The proof of (5) and (6).

If ξ is a non-negative random variable, with expected value M and variance σ^2, that is

$$\int_0^\infty f(x)dx = 1, \quad \int_0^\infty xf(x)dx = M, \quad \int_0^\infty (x - M)^2 f(x)dx = \sigma^2$$

and

$$\int^\infty x^2 f(x)dx = M^2 + \sigma^2.$$

Let the load function $xf(x)$ (Figure 7.)
Moment equilibrium equation for point A is:

$$B(\lambda)\lambda M = \int_0^\infty x\big(xf(x)\big)ax = \int_0^\infty x^2 f(x)dx = M^2 + \sigma^2,$$

$$B(\lambda) = \frac{M^2 + \sigma^2}{\lambda M}.$$

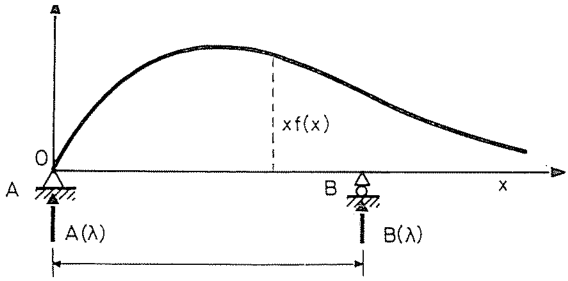

Fig. 7

The reaction force $B(\lambda)$ cannot be less than the external load acting on the cantilever

$$\frac{M^2 + \sigma^2}{\lambda M} \geq \int_0^\infty xf(x)dx \geq \lambda M \int_0^\infty f(x)dx,$$

$$\frac{M^2 + \sigma^2}{\lambda^2 M^2} \geq 1 - F(\lambda M).$$

$$\left(1 + \left(\frac{\sigma}{M}\right)^2\right) \frac{1}{\lambda^2} \geq P(\xi \geq M).$$

References

1. Prekopa, A.: Valószínűségelmélet. Műsz. Könyvkiadó, Bp. 1962.
2. Rényi, A.: Valószínűségelmélet. Tankönyvkiadó. Bp. 1954.
3. Radhakrisma Rao, C.: Lineare Statistische Methoden und ihre Anwendungen. Akademie Verlag, Berlin 1973.
4. Sebestyén, L.: Markov's Inequality in the Case of Random Variable of Concave Distribution. Period. Politechn. *29*, 225 (1985).
5. Sebestyén, L.: Tschebiche's Inequality in the Case of Random Variable of Special Distribution. Period. Polytechn. *29*, 233 (1985).
6. Barabás, B.—Sebestyén, L.: Goodness of Fit with the Help of a Sharpened Bernoulli Inequality. Period. Polytech. *30*, 139 (1986).

Dr. Lukács Sebestyén H-1521 Budapest