CONTRIBUTION TO THE THEORY OF STOCHASTIC PROCESSES

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Abstract

Natural-parameter road surface sample P.S.D. functions give realistic ranking of road quality parameters. There are bounded type variants to the normal and exponential probability functions and boundedness can be calculated from the spectrum. We can define a so-called regular instationary class of stochastic processes.

Introduction

Research reported on in this paper was motivated by the desire to give more correct and detailed description of stochastic environmental influences on moving vehicles. In order to improve calculation and simulation methods a combined approach seemed to offer most promise. Natural parameter analytical P.S.D. formulae, introduction of the sample spectrum concept, bounded type probability distribution/density functions as well as definition of the so-called regular instationary class of stochastic processes has been tried.

Definitions and processes not specified in detail are those given by Bendat and Piersol in Ref. [1].

Notation

f	frequency	1/s
$f(t), g(t)$	function	
n	wave number	1/m
p	probability density	
t	time	s
x	input variable	
x^*	upper boundary	
y	output variable	

$G(f)$	power spectral density function	
$\mathbf{H}_{yx}(f)$	frequency response matrix	
$\mathbf{K}$	damping matrix	Ns/m
L	(integral) scale parameter	m
$\mathbf{M}$	mass matrix	kg
P	probability distribution	
R	correlation/covariance function	
S	measuring/evaluation base length	m
$\mathbf{S}$	stiffness matrix	N/m
V	vehicle speed	m/s, km/h
α, γ	exponent	
δ	boundary ratio	
ζ	space lag	m
λ	Taylor's scale parameter	m
ξ	coordinate parallel to speed	m
σ	standard deviation	
τ	time lag	s
φ	phase angle	rad
ω	circular frequency	rad/s
Φ	shape function	
Ω	space frequency	rad/m

Superscripts:

T	transpose
*	complex conjugate

1. Power spectrum analysis

1.1 Analytical P.S.D. functions in terms of the natural parameters

Finite base-length power spectra of most environmental vehicle load sources have a characteristic negative power-law type appearance for higher frequencies (Fig. 1). In characterizing the process there is little use of following every little local peak or fold on the graph; instead of this smoothing by an appropriate analytical function can be recommended.

For road/rail work there are several functions giving acceptable formal fit over the measured frequency range. Some authors are using simple negative power-law type functions, others are working with polynomial quotient expressions. Some years ago the author proposed the scale parameter concept

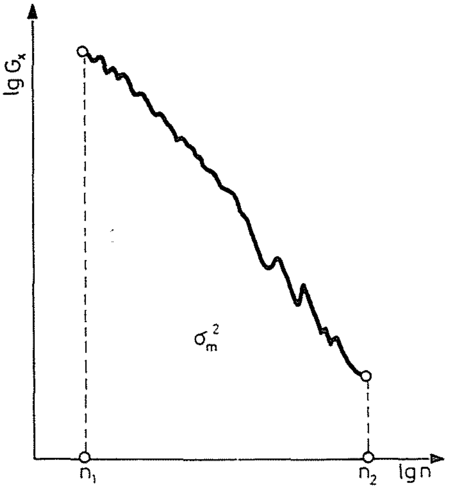

Fig. 1. Truncated spectrum

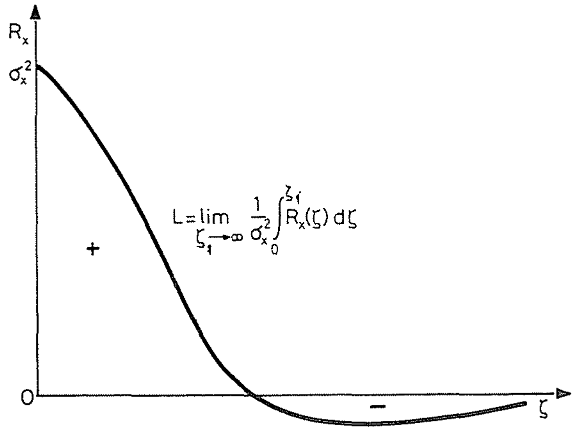

Fig. 2. Definition of the integral scale parameter

for this purpose (See Refs. [4, 5] and Fig. 2). From among the benefits offered by its use the determination of spectrum truncation errors and the possibility of a correct and fast direct spectrum space-time conversion may be emphasized. Mathematical proof for this idea has been given by Kovásznay [3].

Generally speaking, it may be said that the best choice of analytical spectrum functions are those written in terms of the so-called natural parameters. A full list of them as well as the appropriate calculation formulae are given in Table 1.

Table 1

Natural parameters of stationary stochastic processes

Parameter	Calculation formula using $x(\xi)(\bar{x}=0)$	autocorrelation function $R_x(\xi)$	spectral density function $G_x(n)$ resp. $G_x(\Omega)$
Standard deviation σ_x	Def.: $\sigma_x = \lim_{S\to\infty}\left[\frac{1}{S}\int_0^S x^2(\xi)d\xi\right]^{1/2}$		$\sigma_x^2 = \int_0^\infty G_x(n)\,dn = \int_0^\infty G_x(\Omega)\,d\Omega$
Scale parameter (integral) L		Def.: $L = \lim_{\zeta_1\to\infty}\frac{1}{\sigma_x^2}\int_0^{\zeta_1} R_x(\zeta)d\zeta$	Regression analysis
Taylor's scale parameter λ		Def.: $\lambda = \frac{\sqrt{2}\sigma_x}{\left[-\left(\frac{d^2R_x(\zeta)}{d\zeta^2}\right)_{\xi=0}\right]^{1/2}}$	$\lambda = \frac{\sigma_x}{\sqrt{2\pi}}\left[\int_0^\infty n^2 G_x(n)dn\right]^{-1/2}$ $\lambda = \sqrt{2}\sigma_x\left[\int_0^\infty \Omega^2 G_x(\Omega)d\Omega\right]^{-1/2}$
Exponent α			Regression analysis

$$R_x(0) = \sigma_x^2 \qquad G_x(n):\quad G_x(0) = 4L\sigma_x^2 \quad n_{\max} = \frac{1}{\lambda}$$

$$G_x(\Omega):\quad G_x(0) = \frac{2}{\pi}L\sigma_x^2 \quad \Omega_{\max} = \frac{2\pi}{\lambda}$$

Kovásznay [3] has also proved that the zero value of every (one-sided) autospectrum function has to be:

$$G_x(0) = \frac{2}{\pi}\sigma_x^2 L \quad \left(G_x = G_x(\Omega)\right) \tag{1a}$$

being equivalent to

$$G_x(0) = 4\sigma_x^2 L \quad \left(G_x = G_x(n)\right). \tag{1b}$$

It is therefore practical to write spectrum functions in the form:

$$G_x(n) = G_x(0)\,\Phi(n, L, \lambda, \alpha). \tag{2}$$

For road/rail work we are using at present:

$$G_x(n) = \sigma_x^2 \frac{4L}{1 + \left(\frac{4}{\alpha - 1} Ln\right)^\alpha}. \tag{3}$$

1.2 *Process- and sample spectra*

Input time functions for vehicle service load simulation are generated from the P.S.D. function. In case of a four-wheeled vehicle we need four individual representations originating from the same spectrum and having correct relative phase angles. Unfortunately, customary P.S.D. graphs are lacking any information as regards the phase angle. After Rayleigh we define the auto-spectrum as the Fourier transform of the autocorrelation/autocovariance function having thus — and by definition — zero phase angle for all frequencies. Relative phase angles can be expressed only by means of cross-spectra. Input and output auto- and cross-spectra can be arranged in spectral matrices connected by the frequency response matrix $\mathbf{H}_{yx}(f)$ as follows:

$$\mathbf{G}_{yy}(f) = \mathbf{H}^*_{yx}(f)\, \mathbf{G}_{xx}(f)\, \mathbf{H}^T_{yx}(f). \tag{4}$$

The spectrum matrix is really a stopgap arrangement form unsuited for calculation of wheel displacements $x_i(t)$. It is therefore usual to assign random phase angle values to the Fourier components for calculating simulation inputs. Although errors made this way tend to equalize themselves statistically, a better method would be highly desirable. Fortunately, basic features of Fourier series theory indicate that phase angles of every finite-length representation have to increase proportionally to the frequency f resp. n. This theorem, besides giving more realistic phase values, makes also possible to discard the spectral matrix form for theoretical work by switching over to the complex auto-spectrum vector $\mathbf{G}_x(f)$. In this concept Eq. (4) reads:

$$[\mathbf{G}_y(f)]^{1/2} = \mathbf{H}_{yx}(f)\, [\mathbf{G}_x(f)]^{1/2}. \tag{5}$$

This way substantial computer time and space savings are realizable. Full theoretical legalization of the new method may be achieved in the following way.

The classical auto-spectrum $G_x(f)$ — i.e. a spectrum $G_{ii}(f)$ in the diagonal of the spectral matrix $\mathbf{G}_{xx}(f)$ — is in fact a (real type) ensemble spectrum of the process $\{x(t)\}$. Supplemented by the appropriate phase angle $\varphi(f)$ it transforms into a (complex) individual sample spectrum.

1.3 Spectrum parameter ranking with a linear vehicle model

To illustrate the aforementioned by an example, driver vertical acceleration standard deviation σ_a on a bus going with a speed V over three different roads was calculated. The spatial, 8 degree of freedom bus model was characterized by its mass matrix **M**, damping matrix **K** and stiffness matrix S respectively. Results are shown in Fig. 3.

Graph a) is for a smooth cobbled road (Ref. [2] spectrum G; $\sigma_x = 2.245$ cm, $L = 2.91$ m, $\alpha = 2.844$). Points b) are calculated for a black top surface (Ref. [2] spectrum E; $\sigma_x = 3.2$ cm, $L = 45$ m, $\alpha = 2.068$). The short graph c) belongs to an earth road (Ref. [8] spectrum 8; $\sigma_x = 5.783$ cm, $L = 15$ m, $\alpha = 2.463$). Inspection of the graphs reveals two peculiarities.

First, driver acceleration, increasing at first sharply with speed, attains a maximum and then decreases slightly again.

As for the second, a comparison of graphs a) and b) indicates a seemingly contradictory state of affairs. We usually regard σ_x as the measure of road quality and yet road *b*) of $\sigma_x = 3.2$ cm shows off much better than road a) of $\sigma_x = 2.245$ cm. Explanation for this "reversal" is that the asphalt road has a much longer scale length L.

A theoretical survey using the same bus model over synthetic roads of systematically varied parameters has been arranged. A base road having $\sigma_x = 2$ cm, $L = 10$ m and $\alpha = 2$ has been chosen.

Variation of the standard deviation σ_x gave results as shown in Fig. 4. Upper graph points (squares) are for the maximum acceleration σ_a for all

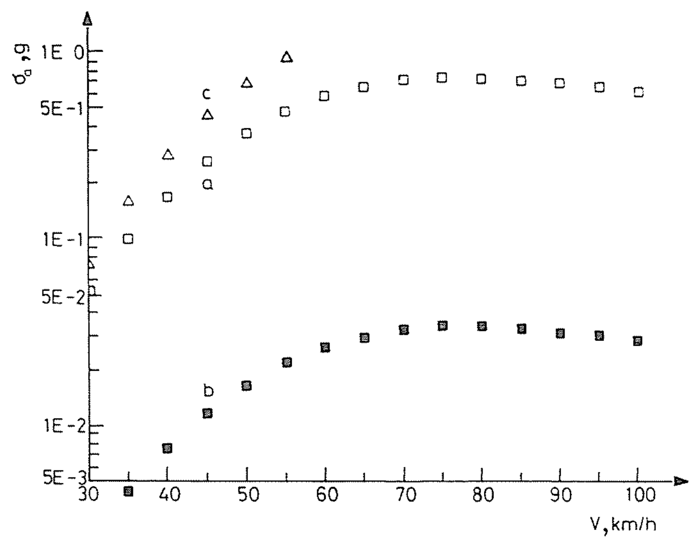

Fig. 3. Driver comfort as function of speed on three different roads

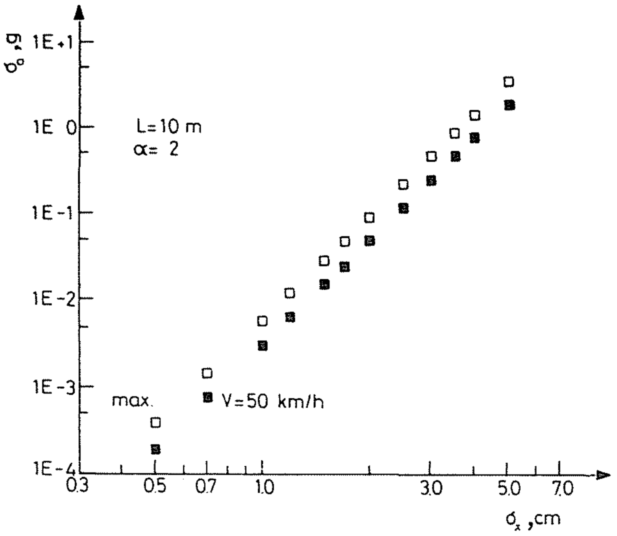

Fig. 4. Influence of σ_x on driver comfort

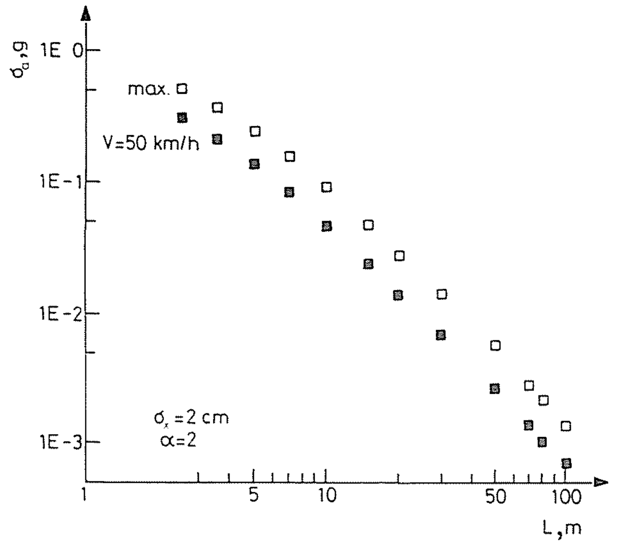

Fig. 5. Influence of L on driver comfort

speeds while the lower ones (full squares) represent results at $V = 50$ km/h in urban traffic driving. Both sequences of results are fitting well to a power function regression with exponent 4. Relative error standard deviations for the regression are 1.273×10^{-3} and 1.816×10^{-3}, respectively. However, this is not to be taken as a physical law, it serves only for having a measure of parameter influence.

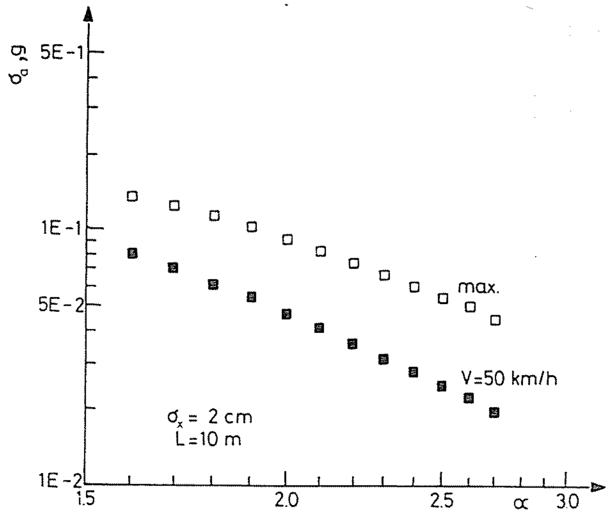

Fig. 6. Influence of α on driver comfort

Results of scale length (L) variations yield graphs in Fig. 5. The influence of this parameter, too, is considerable. It can be approximated by exponents of 1.6 and 1.65, respectively.

Variation of the road spectrum exponent α gave points in Fig. 6. Regression would give for them negative power exponents around 2.1 and 2.7. Nevertheless, we have to rank this parameter as the third one because of its inherently restricted range of variation.

In short, ride smoothness and in consequence road quality, depends not only on the surface roughness standard deviation, all three parameters have to be taken into account.

2. Bounded probability functions

2.1 Coordinate transformations for boundary generation

Because of the finite energy of our systems correct statistics should be of the bounded type. Yet, all functions used for probability assessment are invariably of the ideal, unbounded type. In automotive vehicle design this is not a freak theoretical problem, if misinterpreted, it may have serious economic consequences. So it has been investigated if and how bounded probability functions of exponential resp. normal character can be generated.

As explained in detail in Ref. [6] an upper bound at x^* can be made e.g. by the transformation

$$x \to x \frac{x^*}{x^* - x} = x \frac{\delta}{\delta - \gamma x} \tag{6}$$

writing for the boundary ratio

$$\delta = \gamma x^* \tag{7}$$

The transformed exponential function reads then:

$$P = 1 - \exp\left(- x \frac{\delta}{\delta - \gamma x}\right). \tag{8}$$

Correlation coefficient and other fitting control calculations showed results hardly distinguishable from the unbounded case, at least within the ranges available in normal field work. (See Fig. 7)

A similar transformation may be arranged for the normal (Gauss) statistics having an unbounded standard deviation s. This transformation is using for the boundary ratio

$$\delta = \frac{x^*}{s} \tag{9}$$

and it reads

$$x \to x \frac{x^*}{x^* - x} = \frac{\delta}{\delta - x/s}. \tag{10}$$

The equation giving a bounded normal probability density reads then:

$$p = \frac{C}{\sqrt{2\pi}\, s} \exp\left[- \frac{1}{2}\left(\frac{x}{s} \frac{\delta}{\delta - \frac{x}{s}}\right)^2\right] \tag{11a}$$

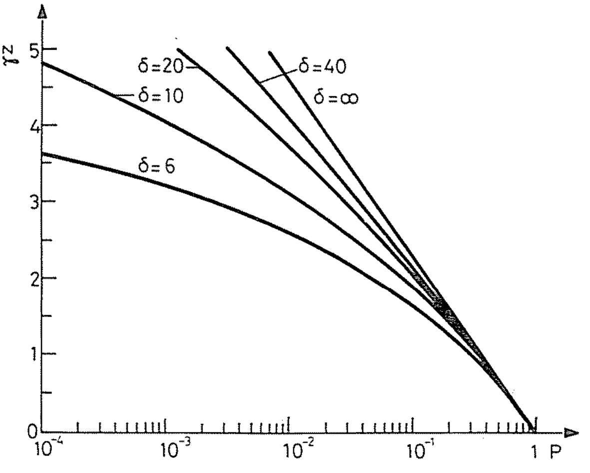

Fig. 7. Bounded type exponential probability distribution functions

with

$$C = \frac{\sqrt{\frac{\pi}{2}}}{\int\limits_0^\infty \exp\left[-\frac{1}{2}\left(\frac{x}{s}\frac{\delta}{\delta-\frac{x}{s}}\right)^2\right] d\left(\frac{x}{s}\right)} . \tag{11b}$$

This distribution, too, is fitting quite well to its ideal predecessor (see Fig. 8).

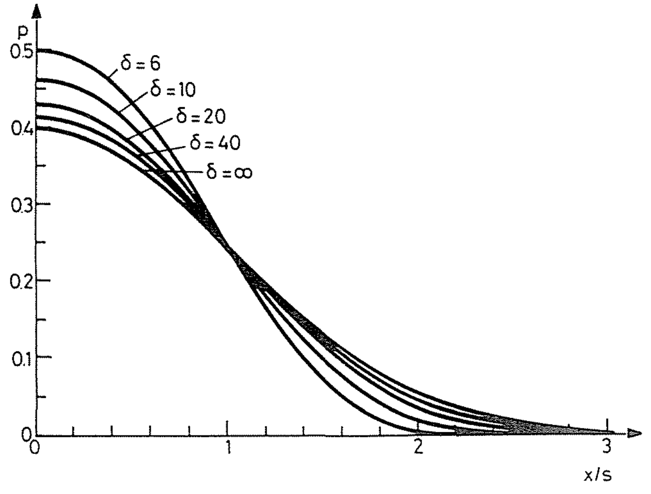

Fig. 8. Bounded type normal (Gauss) probability density functions

2.2 Conditions for the boundedness of the statistics

For want of a reliable method for indicating the boundedness of statistical field data distributions and for the positive proof of the existence of a boundary a search for spectrum criteria has been made.

Starting from basic Fourier-series relations it can be shown that a stationary process is upper bounded if and only if (see Fig. 9):

a) its spectrum is bounded ($G_x(n) \leq C$) and

b) either Taylor's scale parameter $\lambda > 0$, or the exponent of the spectrum $\alpha > 2$.

Particulars of the proof are in Ref. [7].

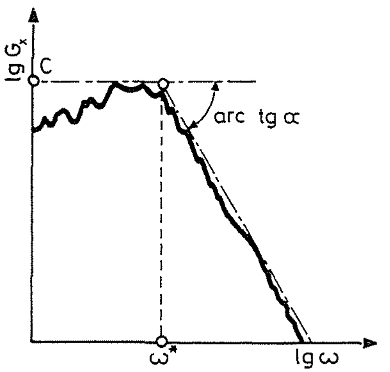

Fig. 9. Conditions for the boundedness of process statistics

3. Regular instationary processes

Research for vehicle dynamics applications is going on concerning instationarities characterized by the stationarity of the transformed representations

$$f(t) \cdot x(g(t))$$

$f(t)$ and $g(t)$ being continuous and smooth functions.

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