SIMULATION MODEL OF GEAR TOOTH STIFFNESS FUNCTION WITH MANUFACTURING ERRORS

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1. Introduction

The mechanical drive systems especially in vehicle applications are subjected to stochastic load conditions, varying in a wide range of load levels and excitation frequencies. The efficient component design needs the detailed knowledge of system answer under service load conditions, even in the stage of preliminary studies and parameter optimalization.

Hence, the numerical dynamic analysis procedures of complex drive systems, in order to optimize element characteristics and service properties are of prime importance [1].

Mechanical transmission systems contain generally a large number of gear trains, which introduce a supplementary excitation caused by a time or angular displacement dependent stiffness variation.

This latter influences on the one hand, the dynamic overloads of the gear box elements and, on the other hand, reacts on the drive and driven system. The importance of this reaction effect is influenced by the stiffness characteristics of the connecting elements.

Furthermore, we present and discuss a model for describing the gear stiffness function taking into account all the important factors and non-linearities.

For the gear transmission stiffness description it is sufficient to deal with a simple two mass system.

In the model in Fig. 1a the Θ_1 and Θ_2 are the moment of inertia of the pinion and the wheel respectively, φ_1, φ_2 are the angular displacements, r_{b1} and r_{b2} the base circle radii, $T_1(t)$ and $T_2(t)$ the outer torques. The tooth contact elasticity is represented by a parallel connected spring and damping element, applied on the pressure line.

The differential equation can be written in the following form:

$$\mathbf{M}\ddot{\boldsymbol{\varphi}} + \mathbf{K}\dot{\boldsymbol{\varphi}} + \mathbf{S}\boldsymbol{\varphi} + \mathbf{T} = 0 \tag{1}$$

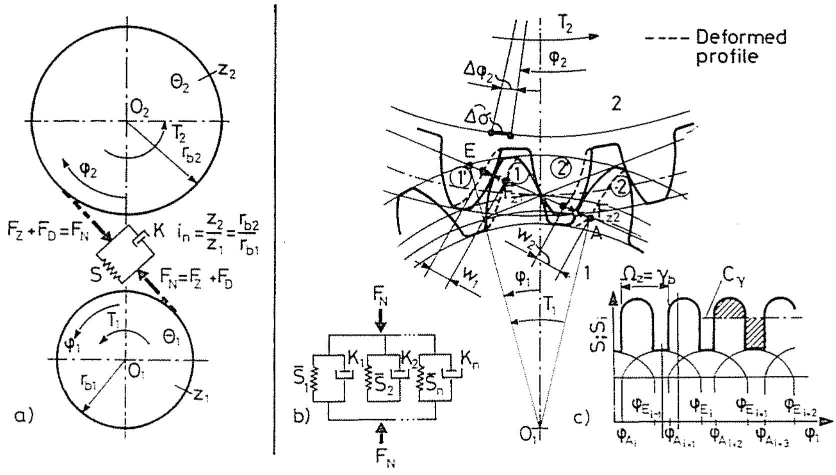

Fig. 1. Dynamic model of the gear train

where $\mathbb{M}$ is the mass matrix, $\mathbb{K}$ is the damping matrix, assuming a speed dependent damping characteristic, $\mathbb{T}$ is for the outer loads, and $\ddot{\varphi}$, $\dot{\varphi}$ are the time derivatives of the angular desplacement vector.

The $\mathbb{S}$ stiffness matrix, is the expanded form of

$$\mathbb{S} = s\begin{vmatrix} r_{b1}^2 & -r_{b1}\, r_{b2} \\ -r_{b1}r_{b2} & r_{b2}^2 \end{vmatrix}; \quad \varphi = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}. \tag{2}$$

The term $\mathbb{K}\dot{\varphi}$ gives the damping torque. The elastic torque component of the total torque is produced by the elastic force:

$$F_{b_z} = s(r_{b1}\varphi_1 - r_{b2}\varphi_2) \tag{3}$$

where s is the stiffness of the spring, and

$$\Delta\sigma = (r_{b1}\varphi_1 - r_{b2}\varphi_2) = r_{b2}(1/i_n\varphi_1 - \varphi_2) = r_{b2} \cdot \Delta\varphi_2$$

$$\Delta\varphi_2 = \varphi_{2n} - \varphi_2;\ \varphi_{2n} = 1/i_n \cdot \varphi_1 \tag{4}$$

i_n the nominal ratio, φ_{2n} the nominal angular position of the wheel, the $\Delta\varphi_2$ by the tooth spring deformation introduced agular error, relative to the nominal position, and $\Delta\sigma$ the travel error measured on the pressure line, equal to the actual tooth deflection.

2. Spring system model of the elastic tooth contact

Considering the real tooth contact, the model is based on a parallel connected spring system, Fig. 1b, where $\bar{s} = \bar{s}(\varphi_1)$ is the single tooth contact stiffness, depending on the contact point position [2]. Having $j = 1 \ldots N(\varphi_1)$ teeth in mesh, the resultant stiffness related to the actual travel error is:

$$s(\varphi_1) = \{\sum_{j=1}^{N} \bar{s}_j(\varphi_1)\, w_j\}/\Delta\sigma \qquad (5)$$

where w_j is the j-th contacting tooth pair deflection.

So, during the rotation, $s(\varphi_1)$ varies as well, because of the variation of $s(\varphi_1)$ and the number of tooth pairs in mesh, $N(\varphi_1)$. In the case of errorless teeth and not considering the tooth mesh troubles caused by tooth deflection and wheel rim deformations $\Delta\sigma = w_j$ in Eq. (5), for all j. Assuming linear $\bar{s}$ single tooth stiffness, the stiffness function can be discribed, based on the theoretical tooth geometry, as a single variable periodic function, with the period of the angle $\Omega = 2\pi/z_1$, and at n_1 input speed of rotation, the frequency $f_z = 2\pi/\Omega n_1 = z_1 \cdot n_1$, z_1 the number of teeth of the pinion, and f_z the tooth frequency, Fig. 1c.

The vibration of the gear train based on this simplified model had been studied by several authors (see ex. [3] [4]) for the estimation of the inner dynamic overload factors, in the case of constant outer torques. Because of the parametric excitation, the process is governed basically by the rheonom vibrating phenomenon, presenting more resonance points even in the case of $f_z < f_e, f_e$ the natural frequency of the system [5].

For entering the tooth manufacturing errors, different kinds of error functions have been introduced by several authors [5] [6], in the form of deterministic harmonic functions, or by randomly generated error values, for altering the actual tooth deflections and so the force – travel error relationship, Eq. (3). The so defined error functions, being in weak correlation with the real tooth error excitation, give only an approximate rough qualitative estimation. Especially, in complex system analysis over- or underestimations can occur.

For more exact stiffness variation description, the following factors must be considered:

- The tooth deflection results tooth mesh beginning and end point variations depending on the actual load and errors. On these irregular mesh zones, the ratio, $i(\varphi_1) \neq i_n$, not even for errorless tooth.
- Manufacturing errors cause $i(\varphi_1) \neq i_n$ on the whole mesh and the load distribution on the meshing tooth pairs is disturbed. The condition of $\Delta\sigma = w_j$ in Eq. 5 is not valid.
- Profile corrections result in considerable transmission characteristics improvement which must be handled.
- By direction change of the tooth normal force (caused eventually by vibrations or abrupt outer load change) the backlash zone is

passed with zero load, followed by the back profile mesh, with different stiffness.

- Based on recent results of tooth deflection measurements [7], the $\bar{s}(\varphi_1)$ single tooth contact caracteristics are not linear.
- Friction force effect. (Not treated here.)

For describing the above factors we can introduce a more detailed spring system model, Fig. 2. The parallel connected spring damper elements represent single tooth pairs, being in contact position, and are fixed on the pinion. For forward and reversed torque transmission, the spring characteristics are different, $\bar{s}_j(\varphi_1) = \bar{s}_j'(\varphi_1)$, and the number of meshing tooth pairs is generally different too. In the case of torque direction reverse the $h(\varphi_1)$ backlash is passed at first, with zero force. In both load transmitting directions, for each

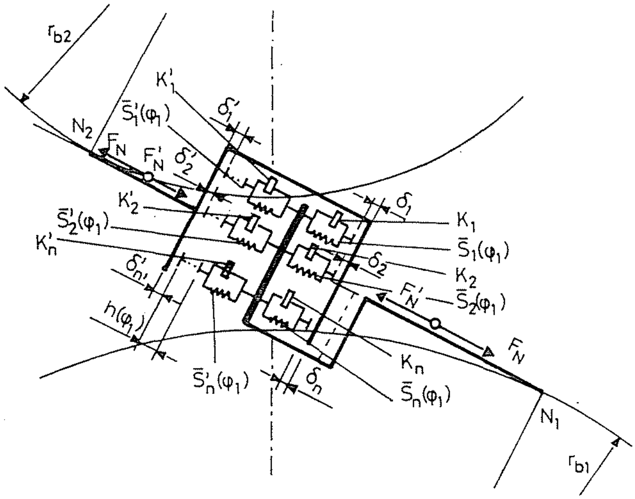

Fig. 2. Tooth contact spring system model

individual tooth pair, the actual fabrication error values, effective in the given contact point present as clearances relative to the ideal contact point position, corresponding to errorless tooth pair. This clearance value, expressed in form of travel error to contact on the pressure line is $\delta_j(\varphi_1)$, depends on the actual φ_1 angular position of the pinion, so for the whole angular contact zone of the j-th tooth pair, it can be given by the contact function, defined as follows:

$$\delta_j(\varphi_1) = r_{b2}(\varphi_{2n} - \varphi_2). \tag{6}$$

For complete characterization of a gear train, the contact function description is needed for all different profile pair combinations.

3. The contact function description [8]

Let us consider the angular interval, where the j-th pair of profiles can enter into contact, Fig. 3. Considering at first errorless profiles and zero outer loads, the theoretical beginning and end mesh points are P and Q. On the mesh interval PA the ratio $i(\varphi_1) < i_n$, and on EQ $i(\varphi_1) > i_n$, not constant. Regular mesh conditions are only on the AE section with $i = i_n$. So, on the PA and EQ irregular mesh areas, if contact takes place, the rotation transmission is not uniform, causing a kinematic excitation, even in the case of ideal teeth. On the irregular zones $\delta_T(\varphi_1) \neq 0$, it can be given by tooth geometry calculations, not detailed here. The errorless contact function is then defined on the interval $\varphi_{1j}^I < \varphi_1 < \varphi_{1j}^S$ and $\delta_T(\varphi_1) = 0$ on the regular AE zone, and $\delta_T \neq 0$ on the rregular PA and EQ zones. Fig. 3.

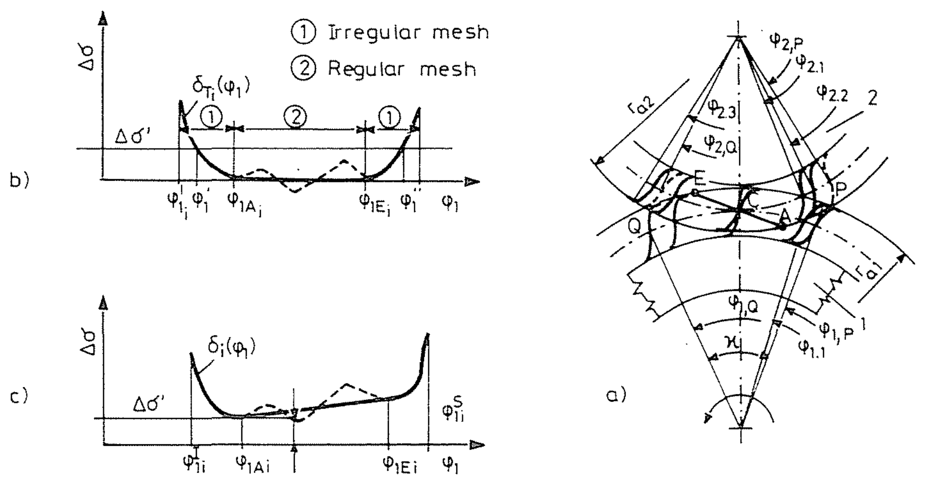

Fig. 3. Contact function for profile pairs

For profiles with manufacturing errors, the contact function derives as transformation of the ideal one. The pitch error results in a simultaneous translation in φ_1 and δ axle directions respectively, while the base circle error can be described by changing the slope of the straight line section on $\varphi_{Aj} < \varphi_1 < \varphi_{Ej}$; The profile form error can be characterized by superimposing a wave form function on the base line. For the sake of brevity, we assume a gear model without form error.

For real gears with z_1 and z_2 number of teeth, $n = z_1 z_2 / k$ elementary contact functions can be given, on the basis of the individual errors, k being the highest common factor of z_1 and z_2. The kinematic excitation function of a gear train can be derived by the successive composition of the $\delta_j(\varphi_1)$

functions, where the succession order is strongly determined by the tooth succession order. Being $\delta_i(\varphi_1) \neq \delta_j(\varphi_1)$ the period of the function $\Omega = 2\pi z_2/k$. Because of the overlapping of the $\varkappa$ angular intervals, at each φ_1 there is at least one pair of profiles in contact position. Under zero load conditions, the rotation of the driven wheel is governed by the contact function of the gear pair:

$$\delta(\varphi_1) = \min_{j=1..n} \{\delta_j(\varphi_1)\}. \tag{6a}$$

The general character of the contact function is represented in Fig. 4. The low frequency components are produced by the pitch error accumulation variations, while the higher ones by the individual pitch and profile errors. For all pairs of profiles the mesh begins and ends by the irregular mesh zone.

The position of the $\delta_j(\varphi_1)$ function relative to the $\Delta\sigma = 0$ axis depends on the choice of the starting point. In each case, there exists a $0 < \varphi_1^* < \Omega$ for which $\delta(\varphi_1^*) \leq \delta(\varphi_1)$. So, by appropriete transformation, one can have $\delta(\varphi_1) > 0$, so $\delta_j(\varphi_1) > 0$. The so defined ordered series of the elementary contact functions is one representation of all possible contact functions, belonging to real gears. This function contains further the clearance values for all profile pairs and pinion angular positions, for the spring model of Fig. 2 relative to the reference point $(\varphi_1^*, \delta(\varphi_1^*))$. (Fig. 4a.)

In the case of intermittent torque reversing, the backlash $h(\varphi_1)$ is passed before the contact on the opposite profile flanks. For this case, the contact functions $\delta(\varphi_1)$ and $\delta'(\varphi_1)$ can be defined in the same way, and attached to the previous one by the $h(0)$ backlash value, and $\delta(0) = 0$. Let be $a > 0$ the value defined as

$$a = \max_{j=1..n} \{\delta_j'(\varphi_1)\} - h(0) \tag{7}$$

Then the $\delta'; \varphi_1$ coordinate system with origin point in $\Delta\sigma = a, \varphi_1 = 0$ and δ' axis in opposite direction gives $\delta' > 0$ values, Fig. 4b. This type of manufacturing error description permits us to enter into the dynamic calculation all kinds of manufacturing errors, described by the $\delta_j(\varphi_1) > 0$, $\delta_j'(\varphi_1) > 0$ contact function series, representing the clearance values and backlash before entering into contact, measured in driven wheel travel error on the pressure line, related to the reference point.

4. Generation of contact functions [8]

The contact functions $\delta(\varphi_1)$ cannot be measured on real gears. However, the resultant of Eq. 6a is by a constant factor related to the tangential composite error of a pair of gears [10]. This can be determined by experiment and applied sometimes for gear verification. The individual $\delta_j(\varphi_1)$ functions,

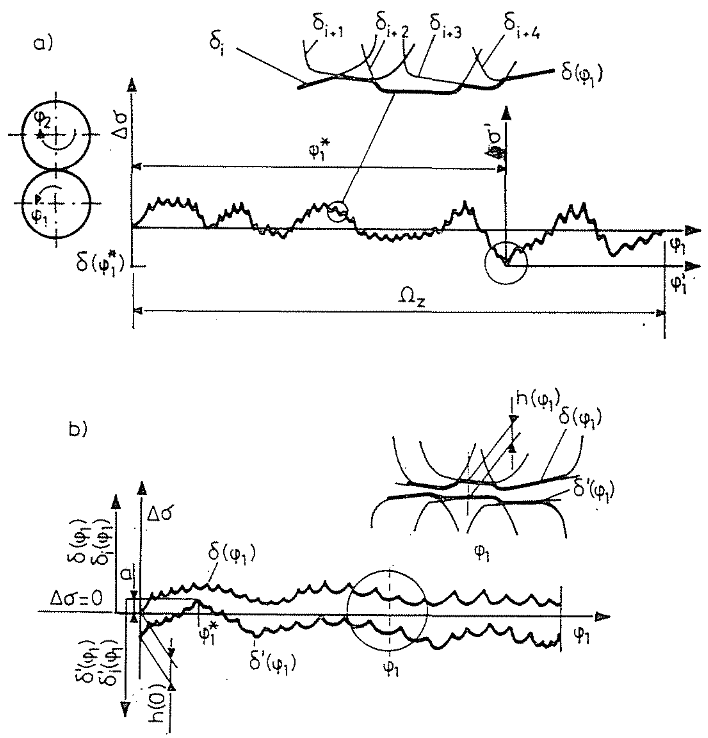

Fig. 4. Gear train contact functions

however, cannot be derived from it. They can only be calculated by the above method based on the individual tooth errors.

Several characteristic values of the tangential composite error function are prescribed in standards by tolerance intervals according to the different gear precision classes. This permits us, in the case of contact function generation, to check the calculated realisation on the basis of standardized values, excluding unrealistic excitation functions.

So, for constructing the contact function series, there are two possible ways:

— for given gears, by individual tooth error measurement one can generate the rolling process to get the ordered series

— assuming the fact that the fabrication errors are of statistical nature, based on the probability density functions of the individual errors, one can by random simulation, generate the real gear model, and simulate the rolling.

In each case, based on the standardized tolerance values, the not allowed realizations can be excluded.

Figure 5 shows the steps of generation, in the case of independent error density functions. At first, gears 1 and 2 are created. This is followed by rolling simulation with master gear, and the resultant contact function is checked. Having two realistic gears, the rolling of them gives the contact function of the service conditions. In this case, further errors as center distance errors, eccentricities etc. can be entered in the simulation. A final checking is the last step for excitation function control.

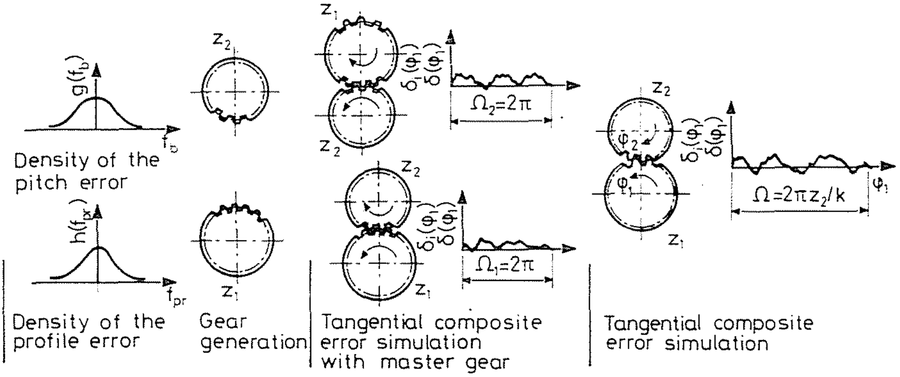

Fig. 5. Scheme of gear generation

5. Generalization of the tooth stiffness function [9]

According to differential equation (1) and the stiffness matrix (2), the stiffness was defined by Eq. (5), the elastic force related to $\Delta\sigma$, the travel error. Maintaining the same structure of the basic mathematical model, one can define the reduced stiffness for gears with manufacturing error, containing all excitation components.

Let us consider one pair of profiles, with manufacturing error, characterized by the $\delta_j(\varphi_1)$ contact function, in the $\Delta\sigma; \varphi_1$ plain, where $\delta_j(\varphi_1) \geq 0$, with reference point according to Eq. 7, Fig. 6a. Let us introduce the indicator function I_j, defined on the $\varphi_{1j}^I < \varphi_1 < \varphi_{1j}^S$ interval, as:

$$I_j(\varphi_1; \Delta\sigma) = \begin{cases} 1 & \text{if} \quad \Delta\sigma > \delta_j(\varphi_1) \\ 0 & \text{if} \quad 0 \leq \Delta\sigma < \delta_j(\varphi_1) \end{cases} . \tag{8}$$

Supposing an actual tooth elastic force F_{zj}, resulting a $\Delta\sigma > 0$ travel error of the wheel,

$$F_{zj} = \bar{s}_j(\varphi_1)\left(\Delta\sigma - \delta_j(\varphi_1)\right) \qquad \Delta\sigma \geq 0 \;;\; \delta_j(\varphi_1) \geq 0. \tag{9}$$

Relating the actual force to the actual travel error:

$$\hat{s}_j(\varphi_1;\ \Delta\sigma) = \bar{s}_j(\varphi_1)\left(\Delta\delta - \delta_j(\varphi_1)\right) I_j(\varphi_1;\ \Delta\sigma)/\Delta\sigma \qquad \hat{s}_j(\varphi_1;\ \Delta\sigma) \geq 0 \tag{10}$$

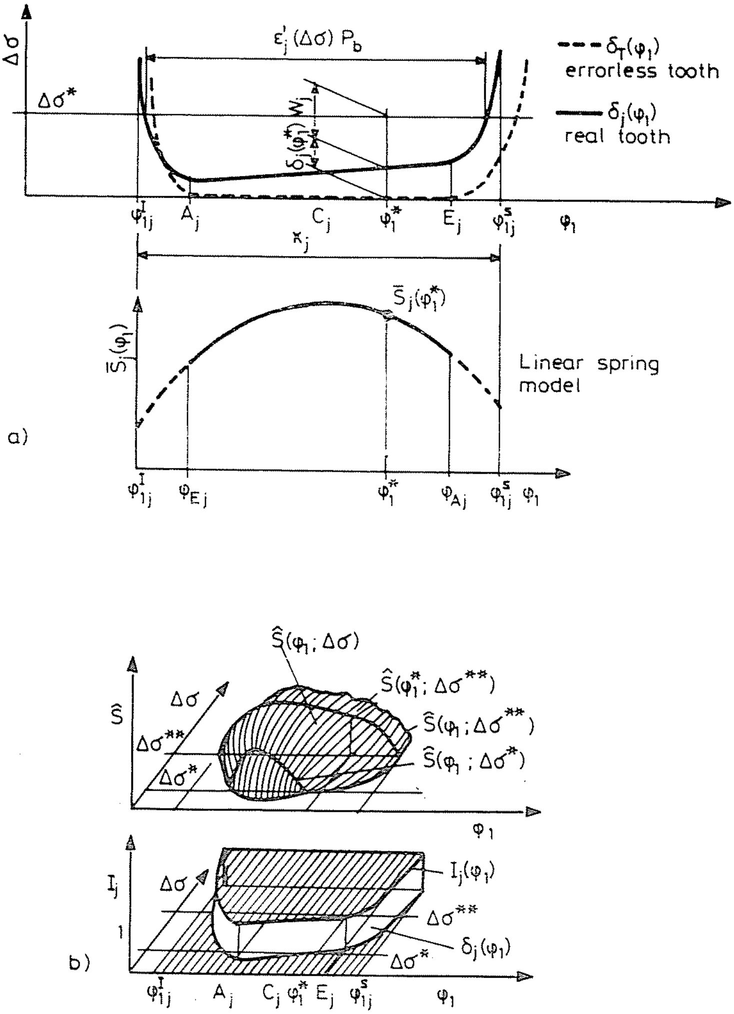

Fig. 6. Reduced stiffness derivation

$\hat{s}(\varphi_1; \Delta\sigma)$ the reduced stiffness of the j-th pair of profiles. In the case of fixed contact function, a two variable function of φ_1 and $\Delta\sigma$, Fig. 6b is obtained. Considering the overlapping of the individual profile pair action intervals, in the case of $j = 1 \ldots N(\varphi_1)$ number of teeth in contact position, the reduced stiffness of the gear train, based on the spring model of Fig. 2 is:

$$\hat{s} = \{\sum_{j=1}^{N} \bar{s}_j(\varphi_1; \Delta\sigma - \delta_j(\varphi_1))\, I_j(\varphi_1; \Delta\sigma)\big(1 - \delta_j(\varphi_1)\big)/\Delta\sigma\}\,. \tag{11}$$

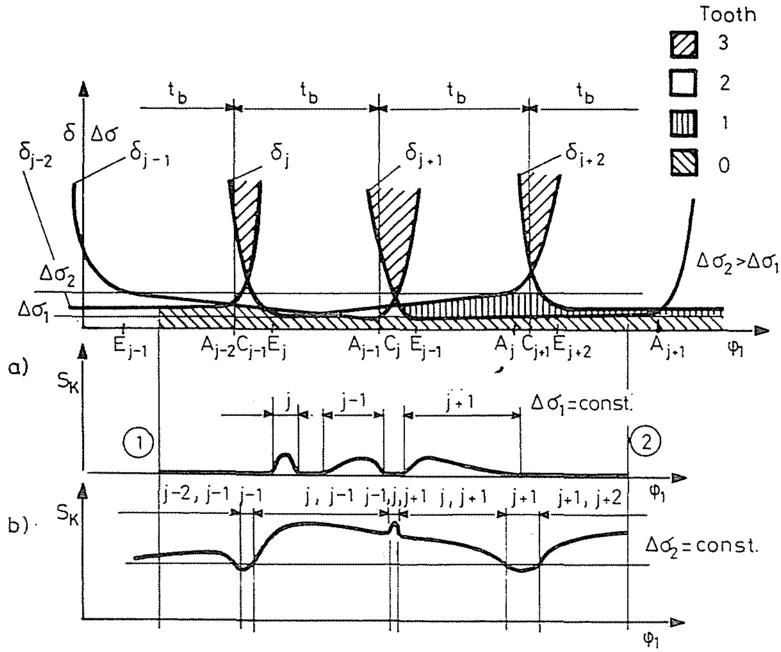

Fig. 7. Reduced stiffness derivation for real mesh

The s containing all the high and low frequency geometrical errors, the irregular mesh effects, the single tooth stiffness variation, the changing of number of teeth in contact, and actual load influence, describes totally the excitation effects introduced by the gear train, Fig. 7.

Taking into account the case of load reversing, the $\hat{s}'$ reduced stiffness for the reversed mesh can be defined in the same way, and preparing the two contact functions by Eq. 7, the general shape of the stiffness function is represented in Fig. 8. So, the s stiffness factor in matrix (2) as a two-variable value, and the displacement vector are as follows:

$$\begin{array}{ll} s = \hat{s}(\varphi_1; \Delta\sigma) & \\ s = 0 & \end{array} ; \boldsymbol{\varphi} = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix} \qquad \begin{array}{c} \Delta\sigma > \delta_j(\varphi_1) \\ \delta_j(\varphi_1) - h(\varphi_1) \leq \Delta\sigma \leq \delta(\varphi_1) \end{array} \tag{12}$$

$$s = \hat{s}'(\varphi_1; \Delta\sigma - a); \boldsymbol{\varphi} = \begin{vmatrix} \varphi_1 - a/2r_{b1} \\ \varphi_2 - a/2r_{b2} \end{vmatrix} \Delta\sigma < \delta(\varphi_1) - h(\varphi_1).$$

This model permits us to take into account the nonlinear operating curve of the single tooth contact stiffness.

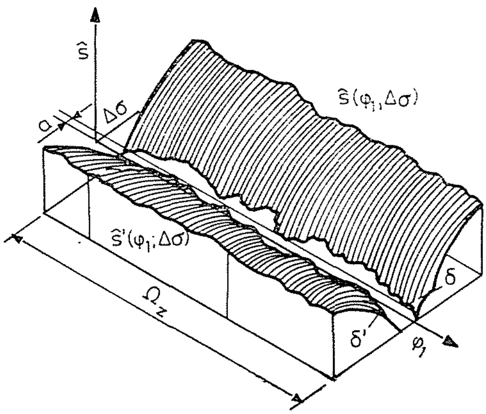

Fig. 8. General shape of the reduced stiffness function

6. Reduced stiffness for normal and modified profiles

On the basis of the reduced stiffness of Eq. (10) all kinds of profiles can be handled. Further more we present some characteristic applications for normal and modified profiles.

6.1 Normal profile

In the case of errorless teeth the reduced stiffness is $s(\varphi_1; \Delta\sigma) = s(\varphi_1)$, s defined by Eq. 5, not depending on the load, Fig. 9a, assuming linear single tooth deflection characteristics. However, even in this case, considering the irregular contact area, only the reduced stiffness can be interpreted, and the load influence alters the stiffness function. For gears with contact ratio near 2, at higher loads 3-tooth contact can occur. In Fig. 9b the nominal contact ratio $\varepsilon = 1.8$, and as higher loads the curve shape is significantly altered. In the case of nonlinear $\bar{s}_j$ single tooth stiffness the curve shape is changed as well, because of the smoother load pick-up of the real teeth, Fig. 9c.

For profiles with error, only the reduced stiffness can be given, resulting in an irregular function variation. In Fig. 10, the same gear contact is represented with linear $\bar{s}_j(\varphi_1)$ values, and positive and negative pitch errors. The reference level for $\delta(\varphi_1)$ is taken on the profile pair with negative pitch error. The $\Delta\sigma = \text{const}$ plain sections show important variations, especially for smaller load levels. In the case of increasing load, $\hat{s} \to s$, the error influence becomes relatively smaller, and the vibration characteristics tend to the ideal one.

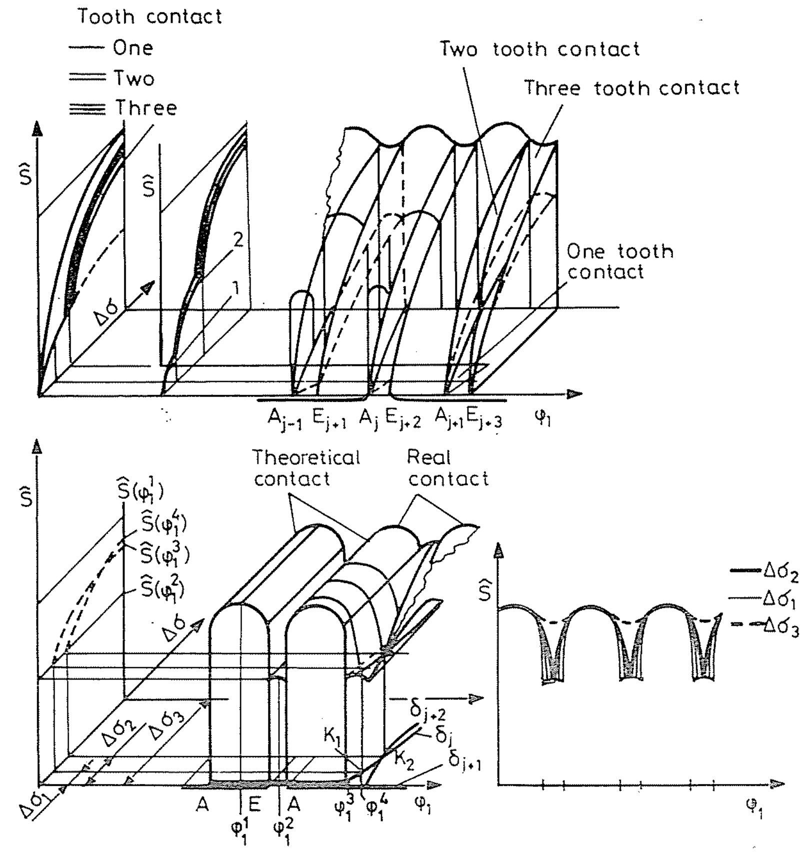

Fig. 9. Stiffness functions for normal profile

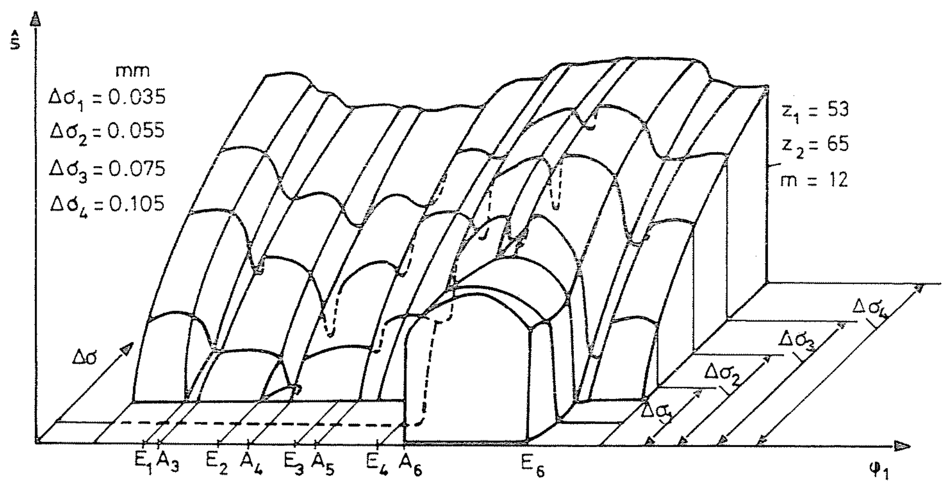

Fig. 10. Stiffness function for gears with pitch error

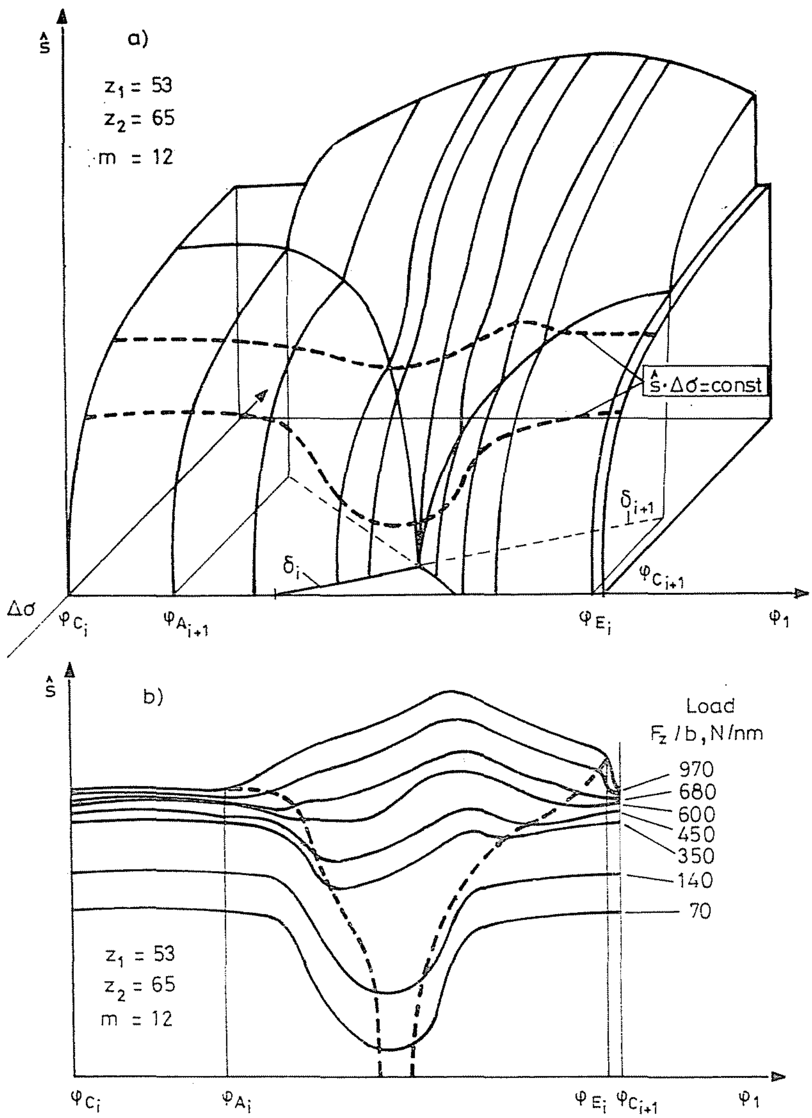

Fig. 11. Stiffness in the case of modified profile

6.2 Modified profile

The modification of the profile is applied for gear transmission characteristics improvement and noise reduction [8]. In this case, for stiffness description, only the reduced stiffness can be applied, taking into account the profile change in the contact functions. Even for errorless teeth, the stiffness function is considerably changed.

In Fig. 11a one period of the stiffness function for errorless teeth is represented, with the same base tooth geometry as previously, but modifications are applied. The load influence is well reflected, and the elevated excitation at lower load levels. In the neighbourhood of the design point, reduced excitation is present. In ideal case, all excitation can be eliminated.

In Fig. 11b the $F_z = \hat{s}(\varphi_1; \Delta\sigma) \cdot \Delta\sigma = \text{const.}$ curves are shown, which are the sections of the stiffness surface with a hyperbolic cylinder. Beside the shape change, the phase is altered too by the load. Fabrication errors introduce further variations, eliminating sometimes the beneficial effects of the modification.

7. Conclusions

For the dynamic analysis of complex drive systems, the gear train introduced excitation description is needed. Introducing a general spring model, all important excitation components can be handled by the analytical description of the ordered series of the contact functions, representing the tooth error and mesh troubles caused nonlinearities. With appropriate choice of the reference level, the reduced stiffness can be defined, containing all excitation effects. The flexibility of the model permits to handle different types of single tooth characteristics.

By contact function generation, real error effects can be simulated and checked on the basis of standardised tolerance values. The model permits further the treatment of modified profiles and the evaluation of their effects on the complex system, operating under variable load conditions, as it is in vehicle applications.

The model discussed is strictly valid only for gears having zero width. For real gears, the alignment error should be considered too. It can be described as a further excitation effect, not detailed here, influencing the reduced stiffness function [9].

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