# APPLICATION OF REDUCED-ORDER LUENBERGER OBSERVER TO THE DESIGN OF ACTIVE SUSPENSION FOR VEHICLES

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Received Aug. 8, 1988.
Presented by Prof. P. Michelberger

## Abstract

This paper investigates the idea of using Luenberger's reduced-order observer for obtaining state estimates of Active Suspension for vehicles. One of the major results presented in this paper is the detailed development of the general solution to the problem of constructing a reduced-order observer and its consequent application to the design of Active Suspension systems.

## 1. Introduction

In the design of Active Suspension for vehicles, we use the common state-space technique for compensation, namely linear state variable feedback (l.s.v.f.). One of the major problems in the implementation of l.s.v.f in the design is that not all the internal states of the suspension system are easily measurable although it should be noted that additional sensors [14] can often be employed to mesure additional components of the states. Generally these sensors are expensive and difficult to implement.

In most of the papers dealing with the design of Active Suspension, eg. [14], [15], [16], [17], [18], [19], [20], [21] the problem of measuring the internal states are but mentioned but not dealt with. In this paper the use of the reduced-order Luenberger observer is proposed as one of the solutions to the problem of internal state measurements. The solution is based on a special linear transformation which transforms the given time-varying continuous state equations into an equivalent state space form which is very convenient from the stand point of observer design. The design of the observer is based on a unique observer configuration containing an arbitrary matrix **L** which arbitrarily positions the eigenvalues of $\mathbf{A} - \mathbf{LC}$ in the half plane $Re(\lambda) < 0$. This matrix **L** can be computed recursively using algorithms similar to Kalmans filter algorithms.

The organization of the paper is as follows. Section 2, deals with the design of Active Suspension systems. Section 3. formulates the design of a reduced-order observer associated with Active Suspension design and Section 4. gives experimental results. The conclusions are stated in Section 5.

## 2. Design of active suspension

A two-degree-of-freedom linear model of a vehicle considered here is given in Fig. 1. $M_s$ represents the sprung mass and $M_u$, the unsprung mass. Absolute vertical displacements of these are $x_1$ and $x_2$ respectively. $k_s$ and $c_s$ denote stiffness and damping ratio of the sprung mass, $k_u$ is the tire stiffness. Suspension forces are supplemented by an active part $\mathbf{u}(t)$, which is a controllable variable. The system is excited by road unevenness $\mathbf{r}(t)$. The passive

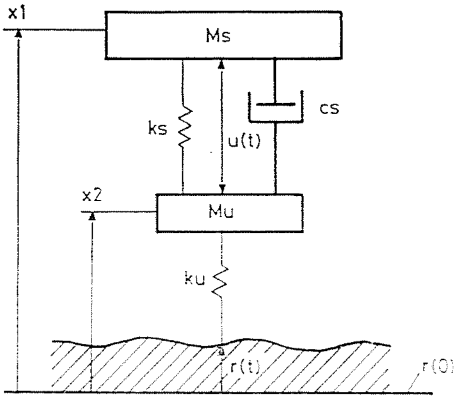


*Fig. 1.*

elements in the suspension were introduced to ensure the vehicle operating in the case of active system failure and to realize a portion of control force which need not be produced by actuator.

The main vehicle responses that are examined are:

1. The vertical acceleration of the sprung mass $(\ddot{x}_1)$
2. The vertical acceleration of the unsprung mass $(\ddot{x}_2)$
3. Suspension deflection $(x_1)$
4. Tire deflection $(x_2)$

If we let $\dot{x}_1 = x_3$, $\dot{x}_2 = x_4$, then the following state differential equations in state space form describe the open loop system (Passive Suspension).

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)\,\mathbf{B}\mathbf{u}(t) + \mathbf{D}\mathbf{r}(t),$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{E}\mathbf{u}(t) + \mathbf{F}\mathbf{r}(t) \tag{1}$$

where the matrices are given by:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s/M_s & -k_s/M_s & -c_s/M_s & -c_s/M_s \\ k_s/M_u & -(k_s+k_u)/M_u & c_s/M_u & -c_s/M_u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1/M_s \\ -1/M_u \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_u/M_u \end{bmatrix} \tag{2}$$

$$\mathbf{C} = \begin{bmatrix} -k_s/M_s & k_s/M_s & -c_s/M_s & c_s/M_s \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_s/M_u & -(k_s+k_u)/m_u & c_s/M_u & -c_s/M_u \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1/M_s \\ 0 \\ 0 \\ -1/M_u \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_u/M_u \end{bmatrix} \tag{3}$$

In the design of active suspension, we need to determine the optimal control force $\mathbf{u}(t)$ for the system described by Eq (1) which minimizes the quadratic performance index $\mathfrak{J}$ given by:

$$\mathfrak{J} = \int_0^\infty [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\mathrm{d}t$$

where $\mathbf{Q}$, $\mathbf{R}$ are appropriately defined weighting matrices.
The optimal control force (control law) $\mathbf{u}(t)$ is given by

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{4}$$

$$\mathbf{K} = -(\mathbf{R} + \mathbf{B}^T\mathbf{P} + \mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} \tag{5}$$

where $\mathbf{K}$ is the steady state solution of Eq (5) and $\mathbf{P}$ is the $n \times n$ symmetric positive definite solution to the algebraic Riccati Equation:

$$\mathbf{P} = \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}(\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} + \mathbf{Q} \tag{6}$$

For more detail of the solution of Riccati's Equation see [11], [12], [13]. By substituting Eq (4) for $\mathbf{u}(t)$ in Eq (1) we get the close- loop system. The control law given by Eq (4) requires availability of all the states $\mathbf{x}(t)$. But, as will be seen, not all the states can easily be measured. We consider now the possibilities to design an observation (measurement) equation associated with the state space model given by Eq (1). Denote by $\mathbf{y}(t)$ the measurement vector, then the measurement

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{E}\mathbf{u}(t) + \mathbf{F}\mathbf{r}(t)$$

where $\mathbf{C}$ is an $m \times n$ $(n = 4)$, and $m$ denotes the number of sensors to be applied. The structure of the row vector in $\mathbf{C}$ depends on the specific measurement situation. A general requirement is that the system given by Eq(1) has to be observable from $\mathbf{y}(t)$.

Choosing the measurements of $x_1$ and $x_2$, the matrix $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$ have the form:

$$\mathbf{C}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{E}_{12} = [0], \quad \mathbf{F}_{12} = [0].$$

It can be deduced, that the system, i.e. the pair $\{\mathbf{A}, \mathbf{C}\}$ is completely observable. The measurements of suspension deflection $(x_1)$ and tire deflection $(x_2)$ e.g. is not a simple problem. It was proposed in [14] that these states could be measured using on ultrasonic transmitter- reciever with laser beams. But this unit is very expensive and difficult to realize.

Another possibility is to apply acceleration measurements. The vertical acceleration of the sprung mass $(\ddot{x}_1)$ can easily be measured using an accelerometer mounted on the sprung mass $M_s$. We could also use the accelerometer to find the vertical acceleration of the unsprung mass. But from the output Eq (3) we see that if we choose $(\ddot{x}_2)$ then the output measurements contains the noise signals.

Considering all the above mentioned facts, the most appropriate output to choose is the vertical acceleration of the sprung mass. In doing so we have reduced our system into a SISO case where the system is given by:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}_3 x(t) + \mathbf{E}_3\mathbf{u}(t) \tag{7}$$

where $$\mathbf{C}_3 = [-k_s/M_s \; k_s/M_s \; -c_s/M_s \; c_s/M_s] \tag{8}$$

and $$\mathbf{E}_3 = [1/m_s]; \quad \mathbf{F}_3 = [0].$$

It can also be shown that the pair $(\mathbf{A}, \mathbf{C}_3)$ is observable. This means that all the other states can be reconstructed from the readings of $\ddot{x}_1(t)$ (accelerometer). The structure of the matrix $\mathbf{C}_3$ comes directly from the choice of the sensor. and the output $\mathbf{y}(t)$ clearly represents the acceleration, $\ddot{x}_1$.

One of the fundamental applications or the observer theory is the design of feed-back controllers for linear regulator problem where some of the states are inaccessible and must therefore be estimated using an observer.

In the design of active suspension we assume that only one of the states can be accurately measured i.e. the vertical acceleration of the sprung mass. The rest of the states are assumed inaccessible. The alternative considered here is to use a reduced-order observer to construct an estimate of the inaccessible states $\tilde{\mathbf{x}}(t)$ and apply the suboptimal feed-back control law

$$\tilde{\mathbf{u}}(t) = -\mathbf{K}\tilde{\mathbf{x}}(t) + \mathbf{G}\mathbf{v}(t) \tag{9}$$

## 3. Reduced-order Luenberger observer and its application to active suspension system

To formulate this problem we consider the following theorem:

*Theorem 1.* (see Wolovich 1974 pp 206).

Consider the system Eq(1).

All $(n)$ eigenvalues of $(\mathbf{A}-\mathbf{LC})$ can be completely and arbitrarily assigned via $\mathbf{L}$ if the pair $(\mathbf{A}, \mathbf{C})$ is observable, any unassignable eigenvalues correspond to the unobservable modes of the system.

From the above theorem we can conclude that if our system Eq(1) is observable, (which can be seen), then we can find the gain matrix $\mathbf{L}$ such that all the $(n)$ eigenvalues of $(\mathbf{A} - \mathbf{LC})$ are located at the left half plane $Re(\lambda) < 0$.

The $n$-dimensional system

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\,\mathbf{C})\tilde{\mathbf{x}}(t + (\mathbf{B} - \mathbf{LE})\mathbf{u}(t) + \mathbf{Ly}(t) \tag{10}$$

is a full order observer for the system Eq(1) if $\tilde{x}(t_0) = x(t_0)$ and $\tilde{\mathrm{x}}(\mathrm{t}) = \mathrm{x}(\mathrm{t})$, $t \geq t_0$, for all $\mathbf{u}(t)$, $t \geq t_0$.
By substracting Eq(10) from Eq(1) we have

$$\dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t) - \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{LCx}(t) - \mathbf{Bu}(t) + \mathbf{LEu}(t) - \mathbf{Ly}(t) \tag{11}$$

or by combining terms in view of (1), that

$$\dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{LC})\big(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\big) \tag{12}$$

In view of the result presented in [6], it is thus clear that

$$\mathbf{x}(t) - \tilde{\mathbf{x}}(t) = \mathrm{e}^{[\mathbf{A}-\mathbf{LC}](t-t_0)}\big[x(t_0 - x(t_0)\big] \tag{13}$$

Comparing Eq(13) and Eq(10), we see that the stability of the observer and the asymptotic behavior of $\mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ are both determined by the structure of the matrix $\mathbf{A} - \mathbf{L}\,\mathbf{C}$. This clearly shows that $\mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ approaches zero, irrespective of its initial value if and only if the observer is asymptotically stable.

If we now let the new control law be given by

$$\mathbf{u}(t) = -\,\mathbf{K}\tilde{\mathbf{x}}(t) + \mathbf{Gv}(t) \tag{14}$$

instead of the actual l.s.v.f control law given by Eq(4) to compensate the given system given by Eq(1) (e.g to attempt to arbitrarily assign all the controllable eigenvalues of the closed-loop system) then by substituting Eq(14) for $\mathbf{u}(t)$ in Eq.(1) and Eq(10) we have

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\tilde{\mathbf{x}}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{BK}\\ \mathbf{LC} & \mathbf{A}-\mathbf{LC}-\mathbf{BK}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \tilde{\mathbf{x}}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{BG}\\ \mathbf{BG}\end{bmatrix}\mathbf{v}(t) \tag{15a}$$

$$\begin{bmatrix}\mathbf{y}(t)\\ \tilde{\mathbf{y}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{C} & -\mathbf{EK}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \tilde{\mathbf{x}}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{EG}\\ \mathbf{0}\end{bmatrix}\mathrm{v}(\mathrm{t}) \tag{15b}$$

If we now transform the Eq(15) via the equivalance transformation

$$\mathbf{Q} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{I} & -\mathbf{I}\end{bmatrix} = \mathbf{Q}^{-1}$$

we obtain the equivalent system:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) - \tilde{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{BG} \\ \mathbf{0} \end{bmatrix} \mathbf{v}(t)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \tilde{\mathbf{y}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C} - \mathbf{EK} & \mathbf{EK} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) - \tilde{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{EG} \\ \mathbf{0} \end{bmatrix} \mathbf{v}(t) \tag{16}$$

From (16) we see that the entire $n$-dimensional "state", $\mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ is not controllable.

Furthermore

$$\left| \lambda\mathbf{I} - \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \right| = |\lambda\mathbf{I} - \mathbf{A} + \mathbf{BK}| \times |\lambda\mathbf{I} - \mathbf{A} + \mathbf{LC}| \tag{17}$$

where we use the notation $|\mathbf{A}|$ for det $\mathbf{A}$. $\mathbf{A}$ is a given matrix. From Eq(17) it is evident that the characteristic polynomial of the overall system is just the product of the characteristic polynomial of the observer and the characteristic polynomial of the suspension system assuming perfect knowledge of the states.

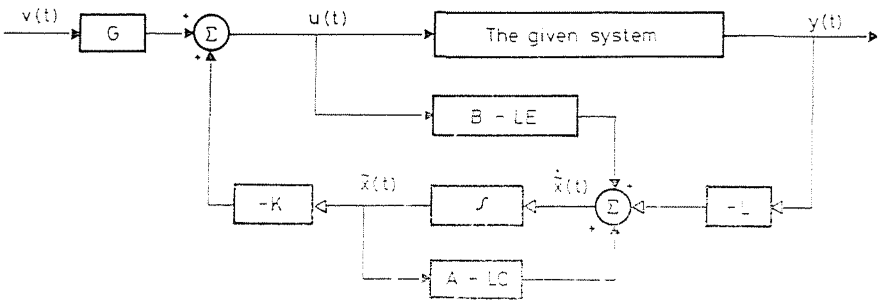


*Fig. 2*

Therefore by theorem (1), we note that given the system Eq(1), and if the system is observable and controllable, then a pair $\{\mathbf{K}, \mathbf{L}\}$ of gain matrices can be chosen to insure complete arbitrary pole assignment and, therefore, the asymptotic stability of the $2n$-dimensional closed-loop system consisting of the given system Eq(1), compensated by an exponential estimator (as depicted in Fig. 2) [6].

Using the matrices in Eq(16) the closed loop transfer function matrix $\mathbf{T}_{K,G,L}(s)$ of the composite $2n$-dimensional system can be found via

$$\mathbf{T}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{18}$$

and is given by:

$$\mathbf{T}_{K,G,L}(s) = [\mathbf{C} - \mathbf{EK} \quad \mathbf{EK}] \begin{bmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{BK} & -\mathbf{BF} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{LC} \end{bmatrix}^{-1} \times \begin{bmatrix} \mathbf{BG} \\ \mathbf{0} \end{bmatrix} + [\mathbf{EG}] \tag{19}$$

or

$$\mathbf{T}_{K,G,L}(s) = \{(\mathbf{C} - \mathbf{EK})(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{BG} + \mathbf{EG}\}$$

Eq(19) is the same as the loop transfer function matrix of the actual l.s.v.f.

To implement Eq(10) as an exponential estimator of the entire state of Eq(1), we need to set up an $n$-dimensional dummy system to approximate the states of the original system. As pointed out in [6] the oder of the observer can actually be less than $n$ because the observed output provides a linear relationship $\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{E}\mathbf{u}(t)$ between the state variables. Therefore it is sufficient to observe $n - p$ of the states and then to calculate the final one from this linear relationship.

We shall consider the scalar output case ($p = 1$) since we only use one sensor i.e. the reading from the accelerometer mounted on the sprung mass $M_s$ and employ the observable companion form introduced in [6]. We shall also follow the steps given in [6].

If we are given an observable system, Eq(7), with $\mathbf{C}_3$ equal to an $n$-dimensional ($n = 4$) raw vector, then we can transform it to an equivalent observable companion form via $\bar{\mathbf{Q}}^{-T}$ (see [6]), where

$$\{\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}_3\, \mathbf{E}_3\} = \{\bar{\mathbf{Q}}^{-T}\mathbf{A}\,\bar{\mathbf{Q}}^{T}, \bar{\mathbf{Q}}^{-T}\mathbf{B}, \mathbf{C}_3\,\bar{\mathbf{Q}}^{T}\mathbf{E}_3\}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 0 & . & . & . & 0 & -a_0 \\ 1 & 0 & . & . & . & 0 & -a_1 \\ . & & & & & . & . \\ . & & & & & . & . \\ . & & & & & . & . \\ 0 & 0 & . & . & . & 1 & -a_{n-1} \end{bmatrix}, \text{ and } \hat{\mathbf{C}}_3 = [0 \;\; 0 \;\; . \;\; . \;\; 1] \tag{20}$$

From the structure of the $\hat{\mathbf{A}}$ matrix it is clear that all $(n)$ eigenvalues of $\hat{\mathbf{A}} - \hat{\mathbf{L}}\hat{\mathbf{C}}_3$ can be completely and arbitrarily assigned via the $n$-dimensional column vector

$$\hat{\mathbf{L}} = [\hat{\mathbf{l}}_1, \hat{\mathbf{l}}_2, \ldots, \hat{\mathbf{l}}_n]^T = \bar{\mathbf{Q}}^T\,\hat{\mathbf{L}} \quad \text{since}$$

$$\hat{\mathbf{A}} - \hat{\mathbf{L}}\hat{\mathbf{C}}_3 = \begin{bmatrix} 0 & 0 & . & . & 0 & -(\hat{l}_1 + a_0) \\ 1 & 0 & . & . & 0 & -(\hat{l}_2 + a_1) \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & 1 & -(\hat{l}_n + a_{n-1}) \end{bmatrix} \tag{21}$$

Which clearly implies that

$$|\lambda\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{L}}\mathbf{C}_3| = \lambda^n + (a_{n-1} + \hat{\mathbf{I}}_n)\lambda^{n-1} + \,.\, + (a_1 + \hat{\mathbf{I}}_2)\lambda + a_0 + \hat{\mathbf{I}}_1 \quad (22)$$

Using Eq(10) we can construct an $n$-dimensional state observer for the system given by Eq(21). Similarily we can also construct a reduced order observer, in our case $(n - p) = 3$, whose state $\tilde{\mathbf{x}}(t) = [\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_{n-1}(t)]^T$ exponentially approaches the $(n-1)$ state variables $\hat{x}_1(t), \hat{x}_2(t), \hat{x}_{n-1}(t)$ (excluding the externally measurable signal $\hat{x}_n(t) = \mathbf{y}(t) - \mathbf{E}_3\mathbf{u}(t)$ of the single output, observable, companion form system:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\,\mathbf{u}(t); \quad (23a)$$

$$\mathbf{y}(t) = \hat{\mathbf{C}}\mathbf{x}(t) + \mathbf{E}_3\mathbf{u}(t). \quad (23b)$$

As outlined in [6] we transform the system into observable companion form using the transformation matrix $\mathbf{P}$ having the form

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & . & . & 0 & -p_0 \\ 0 & 1 & . & . & 0 & -p_1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & -p_{n-2} \\ 0 & 0 & & & 0 & 1 \end{bmatrix} \quad (24)$$

where the $\mathbf{P}_i$ are, as yet, unspecified real numbers. If we now set $\bar{\mathbf{x}}(t) = \mathbf{P}\hat{\mathbf{x}}(t)$, or $\bar{\mathbf{x}}(t) = \mathbf{P}^{-1}\bar{\mathbf{x}}(t)$, it follows that the system

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{u}(t); \quad (25a)$$

$$\mathbf{y}(t) = \bar{\mathbf{C}}_3\bar{\mathbf{x}}(t) + \mathbf{E}_3\mathbf{u}(t), \quad (25b)$$

where $\bar{\mathbf{A}} = \mathbf{P}\bar{\mathbf{A}}\mathbf{P}^{-1}, \bar{\mathbf{B}} = \mathbf{P}\hat{\mathbf{B}}$ and

$\bar{\mathbf{C}}_3 = \hat{\mathbf{C}}_3\mathbf{P}^{-1}$, is equivalent to (23) and therefore to (7) as well. In view of the special form of the transformation matrix, the dynamical equivalent system Eq(25) now assumes a special and rather useful form, since

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & . & . & 0 & p_0 \\ 0 & 1 & . & . & 0 & p_1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & p_{n-2} \\ 0 & 0 & . & . & 0 & 1 \end{bmatrix} \quad (26)$$

it follows that

$$\mathbf{P}\hat{\mathbf{A}}\mathbf{P}^{-1} = \bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & \cdots & -p_0 & -p_0p_{n-2} - a_0 + p_0a_{n-1} \\ 1 & 0 & \cdots & -p_1 & p_0 - p_1p_{n-2} - a_1 + p_1a_{n-1} \\ 0 & 1 & \cdots & -p_2 & p_1 - p_2p_{n-2} - a_2 + p_2a_{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & & & -p_{n-2} & p_{n-3} - p^2{}_{n-2} - a_{n-2} + p_na_{n-1} \\ 0 & & & 1 & p_{n-2} - a_{n-1} \end{bmatrix} \tag{27}$$

$$\mathbf{P}\hat{\mathbf{B}} = \bar{\mathbf{B}} = \begin{bmatrix} b_1 - p_0b_n \\ b_2 - p_1b_n \\ \cdot \\ \cdot \\ b_{n-1} - p_{n-2}b_n \\ b_n \end{bmatrix} \tag{28}$$

$\hat{\mathbf{C}}_3$ and $\mathbf{E}_3$ remains uneffected under the equivalence transformation. If we now denote the first $(n-1)$ components of $\bar{\mathrm{x}}(t)$ by

$$\bar{\mathrm{x}}_{\bar{n}}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \cdot \\ \cdot \\ \bar{x}_{n-1}(t) \end{bmatrix} \quad \text{and define } \bar{\mathbf{B}}$$

as the first $(n-1)$ rows of $\bar{\mathbf{B}}$ then from Eq(25a), Eq(27) and Eq(28) we can obtain a concise state equation for $(n-1)$-dynamical systems with $\mathrm{x}_{\bar{n}}(t)$ as

$$\dot{\bar{\mathrm{x}}}_{\bar{n}}(t) = \bar{\mathrm{A}}_{n-1}\bar{\mathrm{x}}_{\bar{n}}(t) + \bar{\mathrm{A}}_{\bar{n}}\bar{\mathrm{x}}_n(t) + \bar{\mathbf{B}}_{\bar{n}}\mathbf{u}(t) \tag{29}$$

where $\bar{\mathrm{A}}_{n-1}$ is the $(n-1)$ dimensional companion matrix obtained by eliminating both the $n$ — th row and the $n$ — th column of $\bar{\mathrm{A}}$ and $\bar{\mathrm{A}}_{\bar{n}}$ is the column vector consisting of the first $(n-1)$ elements of the last column of $\bar{\mathrm{A}}$.
Noting that

$$\mathrm{y}(t) = \mathrm{x}_n(t) + \mathrm{E}_3\mathbf{u}(t) \tag{30}$$

and by substituting $\mathrm{y}(t) - \mathrm{E}_3\mathbf{u}(t)$ for $\bar{x}_n(t)$ in (29.) the following relationship holds:

$$\dot{\bar{\mathrm{x}}}_{\bar{n}}(t) = \bar{\mathrm{A}}_{n-1}\bar{\mathrm{x}}_{\bar{n}}(t) + \bar{\mathrm{A}}_{\bar{n}}\mathrm{y}(t) + [\bar{\mathbf{B}}_{\bar{n}} - \bar{\mathrm{A}}_{\bar{n}}\mathrm{E}_3]\mathbf{u}(t) \tag{31}$$

If we now construct a dummy $(n-1)$ dimensional system

$$\dot{\tilde{\mathrm{x}}}(t) = \bar{\mathrm{A}}_{n-1}\tilde{\mathrm{x}}(t) + \bar{\mathrm{A}}_{\bar{n}}\mathrm{y}(t) + [\bar{\mathbf{B}}_{\bar{n}} - \bar{\mathrm{A}}_{\bar{n}}\mathrm{E}_3]\mathbf{u}(t) \tag{32}$$

Then
$$\dot{\bar{\mathrm{x}}}_n(t) - \dot{\tilde{\mathrm{x}}}(t) = \bar{\mathrm{A}}_{n-1}[\bar{\mathrm{x}}_{\bar{n}}(t) - \tilde{\mathrm{x}}(t)] \tag{33}$$

and
$$\bar{\mathrm{x}}_{\bar{n}}(t) - \tilde{\mathrm{x}}(t) = \mathrm{e}^{\bar{\mathrm{A}}_{n-1}(t-t_0)}\bar{\mathrm{x}}_{\bar{n}}[(t_0) - \tilde{\mathrm{x}}(t_0) \tag{34}$$

Eq-(32) represents an exponential estimator of Eq(31).

With the right choice of the $(n-1)$ scalars $p_0, p_1, \dots p_{n-2}$ we can position the $(n-1)$ eigenvalues of $\bar{\mathbf{A}}_{n-1}$, which are equal to the zeros of $\lambda^{n-1} + p_{n-2}\lambda^{n-2} + \dots + p_1\lambda + p_0$ in the half-plane $Re(\lambda) < 0$, and from this it is clear that $\tilde{\mathbf{x}}(t)$ will approach $\bar{\mathbf{x}}_{\bar{n}}(t)$ exponentially.

From $\bar{\mathbf{x}}(t) = \mathbf{P}\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{x}}(t) = \bar{\mathbf{Q}}^{-T}\mathbf{x}(t)$, it follows that $\mathbf{x}(t) = \mathbf{P}\bar{\mathbf{Q}}^{-T}\mathbf{X}(t)$, or that the actual state $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \bar{\mathbf{Q}}^T\mathbf{P}^{-1}\bar{\mathbf{x}}(t) = \bar{\mathbf{Q}}\mathbf{P}^{-1}\begin{bmatrix} \tilde{\mathbf{x}}(t) \\ \mathbf{y}(t) - \mathbf{E}_3\mathbf{u}(t) \end{bmatrix} \tag{35}$$

The results obtained for scalar output can be extended to multivariable cases as indicated in [1], [2], [4], [8], [9].

## 4. Experimental results

In numerical calculations the following values for a vehicle model were taken:

$$\begin{aligned} M_s &= 250 \text{ kg;} \\ M_u &= 25 \text{ kg;} \\ k_s &= 5000 \text{ N/m;} \\ c_s &= 250 \text{ Ns/m;} \\ k_u &= 100000 \text{ N/m.} \end{aligned}$$

All the calculations are carried out using PC-MATLAB program. As was indicated before, the design of an observer is possible if the pair $\{\mathbf{A}, \mathbf{C}_3\}$ is completely observable. Using PC-MATLAB we construct the observability matrix $\mathfrak{O} = [\mathbf{C}_3^T \; \mathbf{C}_3\mathbf{A}^T \; (\mathbf{C}_3\mathbf{A}^2)^T \; (\mathbf{C}_3\mathbf{A}^3)^T]$
where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -20 & 20 & -1 & -1 \\ 200 & -4200 & 10 & -10 \end{bmatrix}$$

$$\mathbf{C}_3 = [-20 \quad 20 \quad -1 \quad 1]$$

then the observability matrix becomes

$$\mathfrak{O} = \begin{bmatrix} -20 & 20 & 1 & 1 \\ 220 & -4220 & -9 & 9 \\ 1980 & -37980 & 319 & -4319 \\ -870180 & 18146180 & -41529 & 5529 \end{bmatrix} \tag{36}$$

It can easily be verified that $\mathfrak{O}$ is of full rank and this implies that all the other three states can be reconstructed from the knowledge of the "single" output,

$y_1(t) = (\ddot{x}_1) + E_3u(t)$. To illustrate the procedure outlined in Section 3. for constructing state observers of total dimension 3 ($p = 1$), we must first reduce the system given by Eq(7) to an observable companion form. In reducing the system to an observable companion form we shall use the algorithm outlined in [6].

In particular, we consider the observable system given by Eq(7), or equivalently, the observable pair $\{\mathbf{A}, \mathbf{C}_3\}$ with $\mathbf{C}_3$ of full rank. The dual of the system Eq(7) is readily determined to be the controllable system

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}^T\bar{\mathbf{x}}(t) + \mathbf{C}_3^T\bar{\mathbf{u}}(t) \tag{37}$$

Since $\mathbf{C}_3^T$ is of full rank ($= 4$), it can be reduced to a controllable companion form via an equivalence transformation $\bar{\mathbf{Q}}$.

We can rewrite the system [38] as

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}}\mathbf{x}(t) + \bar{\mathbf{B}}\bar{\mathbf{u}}(t) \tag{38}$$

where $\bar{\mathbf{A}} = \mathbf{A}^T$; $\bar{\mathbf{B}} = \mathbf{C}_3^T$. We can readily verify from the controllability matrix $\bar{\mathfrak{S}}$ that the system is controllable.

$\bar{\mathfrak{S}} = [\bar{\mathbf{B}}, \bar{\mathbf{A}}\bar{\mathbf{B}}, \bar{\mathbf{A}}^2\mathbf{B}, \bar{\mathbf{A}}^3\bar{\mathbf{B}}]$, the matrix consisting of the first $n(= 4)$ linearly independent columns of $\bar{\mathfrak{S}}$.

$$\bar{\mathfrak{S}} = \begin{bmatrix} -20 & 220 & 1980 & -870180 \\ 20 & -4220 & -37980 & 18146180 \\ -1 & -9 & 319 & -41529 \\ 1 & 9 & -4319 & 5529 \end{bmatrix} \tag{39}$$

The transformation matrix $\bar{\mathbf{Q}}$ is obtained from the controllability matrix $\bar{\mathfrak{S}}$ by setting $\bar{q}_1$, the first roaw of $\bar{\mathbf{Q}}$, equal to the last (4—th) row of $\bar{\mathfrak{S}}^{-1}$, and recursively computing the remaining rows of $\bar{\mathbf{Q}}$ by successive postmultiplication of each proceding row of $\mathbf{Q}$ by $\mathbf{A}$. We first calculate $\bar{\mathfrak{S}}^{-1}$

$$\mathfrak{S}^{-1} = 10^{-7*} \begin{bmatrix} -502500 & -25227.27 & -500000 & -45454.54 \\ 25000 & 0 & 527500 & -27500 \\ -56.25 & -5.11 & -1375 & -2397.72 \\ 6.256 & 0.57 & -125 & -11.36 \end{bmatrix} \tag{40}$$

$$\bar{q}_4 = 10^{-7*}\,[6.256\ 0.57\ -125\ -11.361]\ \text{then}$$

$$\bar{\mathbf{Q}} = \begin{bmatrix} \bar{q}_4 \\ \bar{q}_4\bar{\mathbf{A}} \\ \bar{q}_4\bar{\mathbf{A}}^2 \\ \bar{q}_4\bar{\mathbf{A}}^3 \end{bmatrix} = 10^{-7*} \begin{bmatrix} 6.256 & 0.57 & -125 & -11.36 \\ -125 & -11.36 & 0 & -2272.72 \\ 0 & -2272.72 & 0 & 45454.54 \\ 0 & 45454.54 & 0 & 9090909.09 \end{bmatrix} \tag{41}$$

and

$$\bar{\mathbf{Q}}^{-1} = \begin{bmatrix} 0 & -80000 & 0 & -20 \\ 0 & 0 & -4000 & 20 \\ -80000 & -4000 & -20 & -1 \\ 0 & 0 & 20 & 1 \end{bmatrix} \tag{42}$$

Their respective transpose are:

$$\bar{\mathbf{Q}}^{T} = 10^{-7*} \begin{bmatrix} -6.25 & -125 & 0 & 0 \\ 0.57 & -11.36 & -2272.72 & 45454.54 \\ -125 & 0 & 0 & 0 \\ -11.36 & -2272.72 & -45454.54 & 9090909.09 \end{bmatrix} \tag{43}$$

$$\bar{\mathbf{Q}}^{-T} = \begin{bmatrix} 0 & 0 & -80000 & 0 \\ -80000 & 0 & -4000 & 0 \\ 0 & -4000 & -20 & 20 \\ -20 & 20 & -1 & 1 \end{bmatrix} \tag{44}$$

$\bar{\mathbf{Q}}^{T}$ reduces the original system $\{\mathbf{A}, \mathbf{C}_3\}$ to observable companion form given by

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\beta}\mathbf{u}(t); \quad \mathbf{y}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \mathbf{E}_3\mathbf{u}(t) \quad (45a) \quad \text{where}$$

$$\hat{\mathbf{A}} = \bar{\mathbf{Q}}^{-T}\mathbf{A}\bar{\mathbf{Q}}^{T} = \begin{bmatrix} 0 & 0 & 0 & -80000 \\ 1 & 0 & 0 & -4000 \\ 0 & 1 & 0 & -4220 \\ 0 & 0 & 1 & -11 \end{bmatrix} \tag{45b}$$

$$\hat{\mathbf{B}} = \mathbf{Q}^{-T}\mathbf{B} = \begin{bmatrix} -320 \\ -16 \\ -0.88 \\ -0.04 \end{bmatrix} \tag{45c}$$

$$\hat{\mathbf{C}} = \mathbf{C}_3\bar{\mathbf{Q}}^{T} = [0 \quad 0 \quad 0 \quad 0 \quad 1] \tag{46d}$$

Since $\hat{x}_4(t) = y_1(t) - \mathbf{E}_3\mathbf{u}(t)$, we need only estimate $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$. We do this by first employing the equivalence transformation $\mathbf{P}$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{46}$$

If we now set $\bar{\mathbf{x}}(t) = \mathbf{P}\hat{\mathbf{x}}(t)$ or $\hat{\mathbf{x}}(t) = \mathbf{P}^{-1}\bar{\mathbf{x}}(t)$, it follows that the system

$$\dot{\bar{\mathbf{x}}}(t) = \tilde{\tilde{\mathbf{A}}}\bar{\mathbf{x}}(t) + \tilde{\tilde{\mathbf{B}}}\mathbf{u}(t) \tag{47}$$

$$\mathbf{y}(t) = \tilde{\tilde{\mathbf{C}}}x(t) + \mathbf{E}_2\mathbf{u}(t) \qquad \text{where}$$

$$\tilde{\mathbf{A}} = \mathbf{P}\hat{\mathbf{A}}\mathbf{P}^{-1} = \begin{bmatrix} 0 & 0 & -2 & -1851127.16 \\ 1 & 0 & -3 & -41618.73 \\ 0 & 1 & -5 & -4577.70 \\ 0 & 0 & 1 & -31.9 \end{bmatrix} \tag{48}$$

$$\tilde{\mathbf{B}} = \mathbf{P}\hat{\mathbf{B}} = \begin{bmatrix} -319.9 \\ -1587 \\ -0.66 \\ -0.044 \end{bmatrix} \tag{49}$$

is equivalent to Eq(45a).

The matrix $\hat{\mathbf{C}}$ is clearly unaffected by the equivalence transformation $\mathbf{P}$
Let us denote the first 3 components of $\bar{\mathbf{x}}(t)$ by

$$\bar{\mathbf{x}}_3(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad 50)$$

$\bar{\mathbf{B}}_3$ as the first 3 rows of $\tilde{\tilde{\mathbf{B}}}$. If we now define $\bar{\mathbf{A}}_3$ as the 3-dimensional companion matrix obtained by eliminating both the 4-th row and the 4-th column of $\tilde{\tilde{\mathbf{A}}}$ and let $\bar{\mathbf{A}}_4$ represent the column vector consisting of the first 3 elements of the last column of $\tilde{\tilde{\mathbf{A}}}$, we can obtain a concise dynamical equation for the 3-dimensional system with the states $\bar{\mathbf{x}}_3(t)$. It follows that

$$\dot{\bar{\mathbf{x}}}_3(t) = \bar{\mathbf{A}}_3\bar{\mathbf{x}}_3(t) + \bar{\mathbf{A}}_4\bar{\mathbf{x}}_4(t) + \bar{\mathbf{B}}_3\mathbf{u}(t) \tag{51}$$

and since $\mathbf{y}(t) = \bar{\mathbf{x}}_4(t) - \mathbf{E}_3\mathbf{u}(t)$ we obtain by subsituting $\mathbf{y}(t) - \mathbf{E}_3\mathbf{u}(t)$ for $\bar{x}_4(t)$ in Eq(64) the relationship

$$\dot{\bar{\mathbf{x}}}_3(t) = \bar{\mathbf{A}}_3\bar{\mathbf{x}}_3(t) + \bar{\mathbf{A}}_4\mathbf{y}(t) + [\bar{\mathbf{B}}_3 - \bar{\mathbf{A}}_4\mathbf{E}_2]\mathbf{u}(t) \tag{52}$$

We now claim that the following 3-dimensional system is an exponential estimator of Eq(52), i.e.

$$\dot{\tilde{\mathbf{x}}}(t) = \bar{\mathbf{A}}_3\tilde{\mathbf{x}}(t) + \bar{\mathbf{A}}_4\mathbf{y}(t) + [\bar{\mathbf{B}}_3 - \bar{\mathbf{A}}_4\mathbf{E}_3]\mathbf{u}(t) \tag{53}$$

then

$$\dot{\bar{\mathbf{x}}}_3(t) = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix}\bar{\mathbf{x}}_3(t) + \begin{bmatrix} -79988 \\ -3980 \\ -4187 \end{bmatrix}\mathbf{y}(t) + \begin{bmatrix} 0.04 \\ 0.052 \\ 16.088 \end{bmatrix}\mathbf{u}(t) \tag{54}$$

$$\dot{\tilde{\mathbf{x}}}(t) = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix}\tilde{\mathbf{x}}(t) + \begin{bmatrix} -79988 \\ -3980 \\ -4187 \end{bmatrix}\mathbf{y}(t) + \begin{bmatrix} 0.04 \\ 0.052 \\ 16.088 \end{bmatrix}\mathbf{u}(t) \tag{55}$$

It therefore follows that the system given by Eq(55), which can readily be constructed, represents an exponential estimator of $\bar{x}_1(t)$, $\bar{x}_2(t)$, $\bar{x}_3(t)$, since

$$\bar{\mathrm{x}}_3(t) - \tilde{\mathrm{x}}(t) = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix} [\bar{\mathrm{x}}_3(t) - \tilde{\mathrm{x}}(t)] \tag{56}$$

An exponential estimate of the complete system $\tilde{\mathrm{x}}(t)$ can now be obtained from $\bar{x}_1(t)$, $\bar{x}_2(t)$, $\bar{x}_3(t)$ via the equivalence transformation $\mathbf{P}$: i.e.

$$\mathrm{x}(t) = \mathbf{P}^{-1}\mathrm{x}(t) \simeq \mathbf{P}^{-1} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \hat{x}_4 \end{bmatrix} \tag{57}$$

where $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, $\tilde{x}_3(t)$ are the states of the 3-dimensional observer and $\hat{\mathrm{x}}_4(t) = \mathrm{y}_1(t) - \mathbf{E}_3\mathbf{u}(t)$ is a known measurement from the accelerometer which need not be estimated.

## 5. Concluding remarks

By using the feedback of states of a completely controllable and completely observable realization of original state space representation, we can obtain a unew internally stable, minimal-order observer realization whose eigenfrequencies are completely under our control (see Fig. 2).

The application of exponential estimators or Luenberger reduced-order observers in the design of active suspension clearly solves the problem of realizing the optimal state-feed back control since we can obtain and feed back all the internal states as indicated in this work.

However, introducing an observer in the close-loop, generally results in an increase in cost form compared to that obtained when the optimal control law is implimented.

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