THE EQUILIBRIUM EQUATIONS OF THIN-WALLED OPEN SECTION BARS IN TERMS OF CO-ORDINATES OF CENTRE OF GRAVITY

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Received November 6, 1987

Presented by Prof. P. Michelberger

Abstract

To describe the equilibrium equations of thin walled open section bars the co-ordinates of centre of shearing are generally used. On the other hand, the equilibrium equations of closed section bars mostly are described by the co-ordinates of centre of gravity.

Modelling simultaneously open and closed section bars in a framework it is expedient to apply the same type of co-ordinates. For this purpose the co-ordinates of centre of gravity are suitable.

For elaboration of equilibrium equations of thin walled bars in terms of co-ordinates of centre of gravity the principle of total potential energy is applied.

Introduction

Application of closed and open-section bars are common in skeletons and frameworks exposed to dynamic loads (skeletons and frameworks of buildings, technology equipments, vehicle undercarriages, etc.). To describe the equilibrium equations of open section bars the co-ordinates of centre of shearing are generally used. On the other hand, the equilibrium equations of closed section bars can generally be described only by the co-ordinates of centre of gravity. Modelling simultaneously open and closed section bars in a framework, it is necessary to apply the same type of co-ordinates. For this purpose the suitable ones are the co-ordinates of centre of gravity.

The motion (equilibrium) equation can be directly written on mechanical considerations, just as by using the total potential energy functional referring to the given single selected bar. This latter method has noteworthy advantages.

Partly, together with the motion equation, also boundary and initial conditions are obtained so to say automatically that are not simple in this case.

In this paper the total potential energy functional is applied for elaboration of equilibrium equations of thin walled bars in terms of co-ordinates of centre of gravity.

1. The principle of total potential energy

Obviously, the so-called direct generalization of the scalar product (bilinear form) utilized for developing variational principles in linear elasto-statics in the form

$$[\mathbf{u}_1, \mathbf{u}_2] = \int_0^{t_0} \int_V \mathbf{u}_1(\mathbf{x}, t)\mathbf{u}_2(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{1.1}$$

is not symmetric about term $[\partial\mathbf{u}/\partial t, \mathbf{u}]$, therefore operators comprising operand $\frac{\partial \cdot}{\partial t}$ are not of the potential type, hence in general are unsuitable for handling initial conditions of linear elasto-dynamics. [In Eq. (1.1)], V is a single coherent three-dimensional open domain, $o \le t \le t_0$ a confined time interval, while $\mathbf{u}_1(\mathbf{x}, t)$, $\mathbf{u}_2(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$ are quadratically integrable functions.

Again, evidently, scalar product

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \int_0^{t_0} \int_V \mathbf{u}_1(\mathbf{x}, t)\mathbf{u}_2(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{1.2}$$

is symmetric about term $\langle \partial\mathbf{u}/\partial t, \mathbf{u} \rangle$, that is, it perfectly suits development of variational principles of linear elasto-dynamics.

Among relevant research, the most important ones are those due to Gurtin [2, 3], Tonti [4], Oden and Reddy [5] and to Reddy [6, 7]. Gurtin was the first to apply scalar product (1.2) (convolution) for developing linear elasto-dynamic variational principles implicitly containing the initial conditions. Tonti demonstrated scalar product $\langle \partial\mathbf{u}/\partial t, \mathbf{u} \rangle$ to produce a symmetric variational principle referring to the thermal conduction equation.

Variational principles published by Oden and Reddy explicitly contained initial conditions.

In linear elasto-dynamics, like in elasto-statics, variational principles referring to the total potential energy, the complementary energy and the so-called Reissner variational principles are of practical importance. Actually, the principle of total potential energy will be involved, with the following so-called total energy functional:

$$\begin{aligned} \Phi(\mathbf{u}) = \frac{1}{2}\int_0^{t_0}\int_V \varrho(\mathbf{x})\,\dot{\mathbf{u}}(\mathbf{x}, t)\,\dot{\mathbf{u}}(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{1}{2}\int_0^{t_0}\int_V [E(\mathbf{x}) : \varepsilon(\mathbf{x}, t)] : \\ : \varepsilon(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_0^{t_0}\int_V \mathbf{f}(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \\ - \int_0^{t_0}\int_{A_d} \hat{\mathbf{t}}(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_V \varrho(\mathbf{x})\mathbf{v}^0(\mathbf{x})\mathbf{u}(\mathbf{x}, t_0)\,\mathrm{d}\mathbf{x} \end{aligned} \tag{1.3}$$

where:

$0 \leq t \leq t$	confined time interval;
$\mathbf{x} = \mathbf{x}(x, y, z)$	coordinate of place of a point of the given solid;
V	domain occupied by the given solid;
A	boundary (surface) of domain V;
A_d	part of surface A with given surface forces;
$\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$	displacement fector field;
$\varrho = \varrho(\mathbf{x})$	volume intensity of mass distribution;
$\boldsymbol{E} = \boldsymbol{E}(\mathbf{x})$	fourth-order tensor of material characteristics;
$\varepsilon = \varepsilon(\mathbf{x}, t)$	strain vector field;
$\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$	intensity of volume forces;
$\hat{\mathbf{t}} = \hat{\mathbf{t}}(\mathbf{x}, t)$	intensity of prescribed surface forces on surface A_d;
$\mathbf{v}^0 = \mathbf{v}^0(\mathbf{x})$	initial velocity distribution specified for the given solid;
$(.) = \dfrac{\partial(.)}{\partial t}$	(symbol of partial derivation with respect to t);
$:$	symbol of twofold scalar multiplication.

Functional $\emptyset(u)$ involves the following a priori conditions:

$$\sigma(\mathbf{x}, t) = E(\mathbf{x}) : \varepsilon(\mathbf{x}, t) \tag{1.4}$$

$$\varepsilon(\mathbf{x}, t) = \frac{1}{2}\left[\nabla \mathbf{u}(\mathbf{x}, t) + \left(\nabla \mathbf{u}(\mathbf{x}, t)\right)^{\top}\right] \tag{1.5}$$

$$\mathbf{u}(\mathbf{x}, t) = \widehat{\mathbf{u}}(\mathbf{x}, t), \ \mathbf{x} \in A_u, \tag{1.6}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \ \mathbf{x} \in V, \tag{1.7}$$

where:

$\delta = \delta(\mathbf{x}, t)$	stress tensor field;
$\nabla(.) =$	grad (.);
A_u	part of surface A where displacement is given as boundary condition, $A = A_d \cup A_u$, $A_d \cap A_u = \emptyset$ ($\emptyset$ is symbol of an empty set);
$\widehat{\mathbf{u}}(\mathbf{x}, t)$	specified displacement over surface A_u;
$\mathbf{u}^0(\mathbf{x})$	specified initial displacement over domain V.

(1.4) yields the material law, while (1.5) to (1.7) provide for the kinematic possibility of displacement $\mathbf{u}(\mathbf{x}, t)$. Deductions for functional $\Phi(u)$ and for conditions (1.4) to (1.7) are found in [10].

2. Assumptions for writing the motion equation

2.1 Presumed displacement field

The open-section bar has to be modelled as a one-dimensional continuum. The bar is assumed to be prismatic, slender, of a homogeneous, isotropic

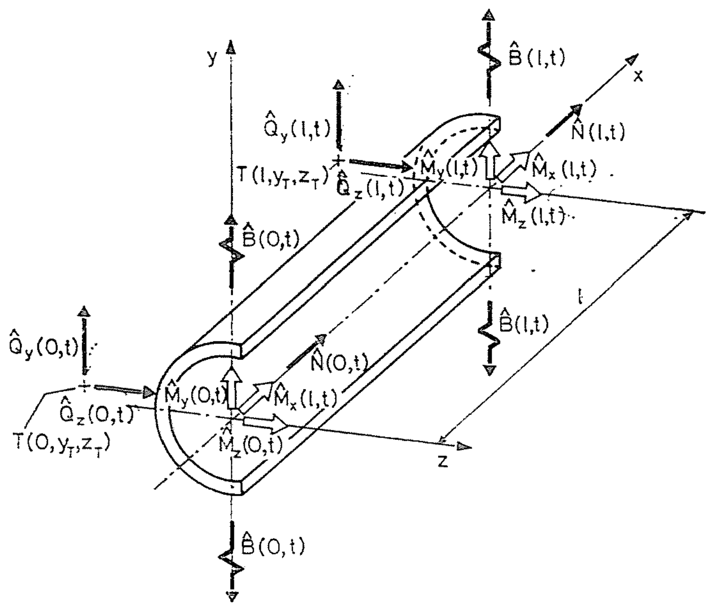

Fig. 1

material. (No assumption of an orthotropic material causes difficulties.) Let the bar be exposed to external forces and moments seen in Fig. 1, and by volume force

$$\mathbf{f}(x, y, z, t)^T = \big(f_x(x, y, z, t),\ f_y(x, y, z, t),\ f_z(x, y, z, t)\big) \tag{2.1}$$

In conformity with symbols in Fig. 1, $\widehat{N}$ is force along the bar, $\widehat{Q}_y$, $\widehat{Q}_z$ are shear forces, $\widehat{M}_x$ the torque, $\widehat{M}_y$, $\widehat{M}_z$ are bending moments and $\widehat{B}$ the so-called bimoment. Axes y and z are assumed to be principal axes of inertia of the bar cross section.

$T(x, y_T, z_T)$ is the torsion center for the cross section of coordinate x.

Displacement of an arbitrary bar point is obtained from

$$\mathbf{u} = \big(u_x(x, y, z, t),\ u_y(x, y, z, t),\ u_z(x, y, z, t)\big) \tag{2.2}$$

$$u_x(x, y, z, t) = u'_{xT}(x, t) - u'_{yT}(x, t)y - u'_{zT}(x, t)z - \varphi'(x, t)\,\omega_T(y, z), \tag{2.2.a}$$

$$u_y(x, y, z, t) = u_{yT}(x, t) - (z - z_T)\,\varphi(x, t), \tag{2.2.b}$$

$$u_z(x, y, z, t) = u_{zT}(x, t) + (y - y_T)\,\varphi(x, t), \tag{2.2.c}$$

where:

$u_{xT}(x, t)$, ... displacement coordinates of an arbitrary point (x, t) of the (straight) torsion axis;

$(\,.\,)' = \dfrac{\partial(\,.\,)}{\partial x}$

$\varphi(x, t)$ angle of rotation of the cross section of coordinate x and normal to the $x - x$ axis about the torsion axis (positive if the vector of rotation points to the positive direction of the x-axis);

$\omega_T(y, z)$ warping rate referred to torsion center T (determined clock-wise on a surface directed by an outer normal unit vector pointing to the negative direction of the x-axis).

2.2 *Conditions for equilibrium equation*

To describe the equilibrium equation by co-ordinates of centre of gravity it is necessary to detail the relationships among the two types of warping characteristics. Denote letter T the torsion (shearing) center and S the gravity center of a cross section of a prismatic bar. Let $\omega_T(y, z)$ warping rate referred to torsion center T and $\omega_S(y, z)$ warping rate referred to gravity center S.

According to denotes of Fig. 2

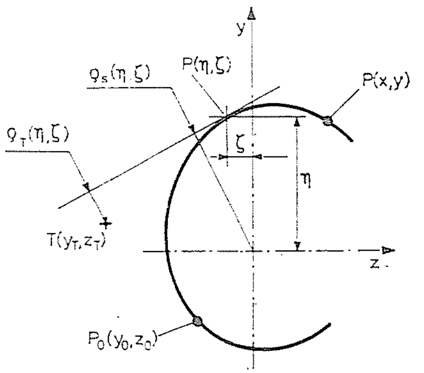

Fig. 2

$$\omega_T(y, z) = \int_{P_0(y_0, z_0)}^{P(y,z)} \varrho_T(\eta, \zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{2.3.a}$$

$$\omega_S(y, z) = \int_{P_0(y_0, z_0)}^{P(y,z)} \varrho_S(\eta, \zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{2.3.b}$$

where $P_0(y_0, z_0)$, $P(y, z)$ are fixed points on curve U. It can be verified that

$$\int_A \omega_T(y, z)\,y\,\mathrm{d}y\,\mathrm{d}z = 0, \tag{2.4.a}$$

$$\int_A \omega_T(y, z)\,z\,\mathrm{d}y\,\mathrm{d}z = 0, \tag{2.4.b}$$

where letter A denotes the area of the cross section of the bar and point $P_0(y_0, z_0)$ is chosen that equation

$$\int_A \omega_T(y, z)\,\mathrm{d}y\,\mathrm{d}z = 0 \tag{2.4.c}$$

is satisfied. Seeing that axes y and z are the main axes of the cross section of the bar the conditions

$$\int_A y\,\mathrm{d}y\,\mathrm{d}z = 0,\ \int_A z\,\mathrm{d}y\,\mathrm{d}z = 0,\ \int_A y\,z\,\mathrm{d}y\,\mathrm{d}z = 0 \tag{2.5.a—c}$$

are satisfied, too.

On the basis of the previous relationships it can easily be seen that

$$\omega_T(y, z) = \omega_S(y, z) - y_T(z - z_0) + z_T(y - y_0). \tag{2.6}$$

Supposing that eq. (2.4.c) is satisfied and one of the axes y and z is symmetry axis of the cross section of the bar. In this case $z_T y_0 - y_T z_0 = 0$ because, or $z_T = 0$ and $z_0 = 0$ (y is symmetry axis) or $y_T = 0$ and $y_0 = 0$ (z is symmetry axis). Using this condition it follows from eq. (2.6) that

$$\omega_S(y, z) = \omega_T(y, z) - z_T y + y_T z \tag{2.7}$$

Utilizing equations (2.5.c) and (2.7):

$$\int_A \omega_S(y, z) y\,\mathrm{d}y\,\mathrm{d}z = -z_T I_{zz}, \tag{2.8.a}$$

$$\int_A \omega_S(y, z) z\,\mathrm{d}y\,\mathrm{d}z = y_T I_{yy}. \tag{2.8.b}$$

where:

$$I_{yy} = \int_A z^2\,\mathrm{d}y\,\mathrm{d}z,\ I_{zz} = \int_A y^2\,\mathrm{d}y\,\mathrm{d}z.$$

According to equations (2.4.c) and (2.5.a—b)

$$\int_A \omega_S(y, z)\,\mathrm{d}y\,\mathrm{d}z = 0. \tag{2.9}$$

Moreover, according to equations (2.4.a—b) and (2.7)

$$I_{\omega_s} = I_{\omega_T} + y_T^2 I_{yy} + z_T^2 I_{zz}, \tag{2.10}$$

where

$$I_{\omega_s} = \int_A \omega_s(y, z)^2\,\mathrm{d}y\,\mathrm{d}z,\ I_{\omega_T} = \int_A \omega_T(y, z)^2\,\mathrm{d}y\,\mathrm{d}z.$$

2.3 *Assumed displacements in terms of co-ordinates of gravity*

After substituting co-ordinates of centre of gravity $(x, 0, 0)$, $0 \leq x \leq l$ for equations (2.2.a—c) the equations

$$u_{xT}(x, t) = u_{xs}(x, t) \tag{2.11.a}$$

$$u_{yT}(x, t) = u_{ys}(x, t) - z_T \varphi(x, t) \tag{2.11.b}$$

$$u_{zT}(x, t) = u_{zs}(x, t) + y_T \varphi(x, t) \tag{2.11.c}$$

are received, where $\omega_T(0, 0) = 0$ by definition. Substituting right hand side of equations (2.11.a—c) for equation (2.2.a—c) and utilizing equation (2.7).

$$u_x(x, y, z, t) = u_{xs}(x, t) - u'_{ys}(x, t)y - u'_{zs}(x, t)z - \omega_s(y, z)\varphi'(x, t), \tag{2.12.a}$$

$$u_y(x, y, z, t) = u_{ys}(x, t) - z\varphi(x, t), \tag{2.12.b}$$

$$u_z(x, y, z, t) = u_{zs}(x, t) + y\varphi(x, t). \tag{2.12.c}$$

2.4 *Boundary and initial conditions*

Surface loads (stresses) specified for bar ends are described by equalities:

$$\hat{\mathfrak{t}}(0, y, z, t)^T = \left(-\hat{\sigma}(0, y, z, t);\ -\hat{\tau}_{xy}(0, y, z, t),\ -\hat{\tau}_{xz}(0, y, z, t)\right), \tag{2.13.a}$$

$$\hat{\mathfrak{t}}(l, y, z, t)^T = \left(\hat{\sigma}(l, y, z, t),\ \hat{\tau}_{xy}(l, y, z, t),\ \hat{\tau}_{xz}(0, y, z, t)\right) \tag{2.13.b}$$

(Negative sign in (2.13.a) refers to the surface of an outer normal pointing to the negative direction.) Stresses $\hat{\mathfrak{t}}(0, y, z, t)$ and $\hat{\mathfrak{t}}(l, y, z, t)$ are assumed to arise as sums of stresses corresponding to elementary ones acting on the bar.

Initial velocity distribution has to be specified according to the assumed distribution field $\mathbf{u}$ (2.12.a—c), that is:

$$\mathbf{v}^0(x, y, z, 0) = \big(v^0_{xs}(x) - v'^0_{ys}(x)y - v'^0_{zs}(x)z - \varkappa'^0(x)\,\omega_s(y, z),$$
$$v^0_{ys}(x) - z\varkappa^0(x),\ v^0_{zT}(x) + y\varkappa^0(x)\big) \tag{2.14}$$

where:

$v^0_{xs}, v^0_{ys}, v^0_{zs}, \varkappa^0, v'^0_{ys}, v'^0_{zs}$ and $\varkappa'^0$ are initial values at time $t = 0$ of velocities and angular velocities $u_{xs}, u_{ys}, u_{zs}, \dot{\varphi}, \dot{u}'_{ys}, u_{zs}$ and $\dot{\varphi}'$ (or of their derivatives with respect to x).

Obviously, also kinematic boundary condition $\hat{u}(x, t)$ and initial displacement $u^0(x)$ in conditions (1.6) and (1.7) have to be specified in conformity with the assumed displacement field (2.12.a—c).

3. Establishment of the motion equation relying in the principle of total potential energy

For the sake of understanding, functional $\varrho(u)$ will be written in the concrete form for the examined problem term wise, and after simplifying notations, each term will be summed in conformity with (1.3).

Expanding term for the kinetic energy by means of Eqs (2.12.a—c), (2.8.a—b) and (2.9):

$$
\begin{aligned}
\Phi(\mathbf{u})_m &= \frac{1}{2}\int_0^{t_0}\int_V \varrho(\mathbf{x})\,\dot{\mathbf{u}}(\mathbf{x},t)\,\dot{\mathbf{u}}(\mathbf{x},t_0-t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \\
&= \frac{1}{2}\int_0^{t_0}\int_0^{l}\int_A \varrho[\dot{u}_{xs}(x,t)-\dot{u}'_{ys}(x,t)y-\dot{u}'_{zs}(x,t)z-\dot{\varphi}'(x,t)\omega_s(y,z)]\,\cdot \\
&\cdot[\dot{u}'_{xs}(x,t_0-t)-\dot{u}'_{ys}(x,t_0-t)y-\dot{u}'_{zs}(x,t_0-t)z-\dot{\varphi}'(x,t_0-t)\omega_s(y,z)]\cdot\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\int_0^{t_0}\int_0^{l}\int_A \varrho[\dot{u}_{ys}(x,t)-z\dot{\varphi}(x,t)][\dot{u}_{ys}(x,t_0-t)-z\dot{\varphi}(x,t_0-t)]\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\int_0^{t_0}\int_0^{l}\int_A \varrho[\dot{u}_{zs}(x,t)+y\dot{\varphi}(x,t)][\dot{u}_{zs}(x,t_0-t)+y\dot{\varphi}(x,t_0-t)]\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t= \\
&= \frac{1}{2}\varrho A\int_0^{t_0}\int_0^{l}\dot{u}_{xs}(x,t)\,\dot{u}_{xs}(x,t_0-t)\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\varrho A\int_0^{t_0}\int_0^{l}\dot{u}_{ys}(x,t)\,\dot{u}_{ys}(y,t_0-t)\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\varrho I_{zz}\int_0^{t_0}\int_0^{l}[\dot{u}'_{ys}(x,t)-z_T\dot{\varphi}'(x,t)]\,\dot{u}'_{ys}(x,t_0-t)\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\varrho A\int_0^{t_0}\int_0^{l}\dot{u}_{zs}(x,t)\,\dot{u}_{zs}(x,t_0-t)\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\varrho I_{yy}\int_0^{t_0}\int_0^{l}[\dot{u}'_{zs}(x,t)+y_T\dot{\varphi}'(x,t)]\,\dot{u}'_{zs}(x,t_0-t)\,\mathrm{d}x\,\mathrm{d}t+ \\
&+\frac{1}{2}\varrho I_{ps}\int_0^{t_0}\int_0^{l}\dot{\varphi}(x,t)\dot{\varphi}(x,t_0-t)\,dx\,dt+\frac{1}{2}\varrho\int_0^{t_0}\int_0^{l}[I_{\omega_s}\dot{\varphi}'(x,t)- \\
&- I_{zz}z_T\,\dot{u}'_{ys}(x,t)+I_{yy}y_T\,\dot{u}'_{zs}(x,t)]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.1}
$$

where:

ϱ constant, mass distribution intensity;

A bar cross section area;

I_{yy} and I_{zz} second-order moments of inertia about axes y and z of the bar cross section;

$$I_{ps} = I_{yy} + I_{zz}.$$

The term for strain energy is expanded to:

$$\Phi(\mathbf{u})_\varepsilon = \frac{1}{2}\int_0^{t_0}\int_V [E(\mathbf{x}) : \varepsilon(\mathbf{x}, t)]\,\varepsilon(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t =$$

$$= \frac{1}{2}E\int_0^{t_0}\int_0^{l}\int_A [u'_{xs}(x, t) - u''_{ys}(x, t)y - u''_{zs}(x, t)z - \varphi''(x, t)\,\omega_s(y, z)]\,\cdot$$

$$\cdot\,[u'_{xs}(x, t_0 - t) - u''_{ys}(x, t_0 - t)y - u''_{zs}(x, t_0 - t)z - \varphi''(x, t_0 - t)\omega_s(y, z)]\cdot dy\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \frac{1}{2}I_T G\int_0^{t_0}\int_0^{l}\varphi'(x, t)\varphi'(x, t_0 - t)\,\mathrm{d}x\,\mathrm{d}t =$$

$$= \frac{1}{2}EA\int_0^{t_0}\int_0^{l} u'_{xs}(x, t)\,u'_{xs}(x, t_0 - t)\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \frac{1}{2}EI_{zz}\int_0^{t_0}\int_0^{l}[u''_{ys}(x, t) - z_T\varphi''(x, t)]\,u''_{ys}(x, t_0 - t)\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \frac{1}{2}EI_{yy}\int_0^{t_0}\int_0^{l}[u''_{zs}(x, t) + y_T\varphi''(x, t)]\,u''_{zs}(x, t_0 - t)\,\mathrm{d}x\mathrm{d}t +$$

$$+ \frac{1}{2}E\int_0^{t_0}\int_0^{l}[-I_{zz}z_T u''_{ys}(x, t) + I_{yy}y_T u''_{zs}(x, t) +$$

$$+ I_{\omega_s}\varphi''(x, t)]\,\varphi''(x, t_0 - t)\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \frac{1}{2}I_{pT}G\int_0^{t_0}\int_0^{l}\varphi'(x, t)\varphi'(x, t_0 - t)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.2)$$

where:

$E = \dfrac{E^*}{1 - \nu^2}$, $G = \dfrac{E^*}{2(1 + \nu)}$, E^* is the Young's modulus; and ν the Poisson's ratio;

I_t — second-order moment of Saint Venant torsion of the bar cross section;

$$\varepsilon_x = \partial u_x(x, y, z, t)/\partial x$$

Term volume force work is expanded by means of (2.1) and (2.12.a—c).

$$\Phi(\mathbf{u}_f) = \int_0^{t_0}\int_V \mathbf{f}(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t_0 - t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^{t_0}\int_0^{l}\left\{\int_A f(x, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{xs}(x, t_0 - t) -\right.$$

$$- \int_A f_x(x, y, z, t)y\,\mathrm{d}y\,\mathrm{d}z\,u'_{ys}(x, t_0 - t) - \int_A f_x(x, y, z, t)z\,\mathrm{d}y\,\mathrm{d}z\,u'_{zs}(x, t_0 - t) -$$

$$- \int_A f_x(x, y, z, t)\,\omega_s(y, z)\,\mathrm{d}y\,\mathrm{d}z\,\varphi'(x, t_0 - t) +$$

$$+ \int_A f_y(x, y, z, t)\,dy\,dz\,u'_{ys}(x, t_0 - t) - \int_A f_z(x, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u'_{zs}(x, t_0 - t) +$$

$$\left. + \int_A [yf_z(x, y, z, t) - zf_y(x, y, z, t)]\,\mathrm{d}y\,\mathrm{d}z\,\varphi(y, t_0 - t)\right\}\mathrm{d}x\,\mathrm{d}t =$$

$$= \int_0^{t_0}\int_0^{l} [q_x(x, t)\,u_{xs}(x, t_0 - t) + q_y(y, t)\,u_{ys}(x, t_0 - t) + m_z(x, t)\,u'_{ys}(x, t_0 - t) +$$

$$+ q_z(x, t)\,u_{zs}(x, t_0 - t) - m_y(x, t)\,u'_{zs}(x, t_0 - t) +$$

$$+ m_x(x, t)\,\varphi_x(x, t_0 - t) - m_{\omega_s}(x, t)\,\varphi'(x, t_0 - t)]\,\mathrm{d}x\,\mathrm{d}t \tag{3.3}$$

introducing simplifying notations for integrals on surface A, with the following meaning:

q_x, q_y, q_z are intensities in directions x, y and z of volume forces acting on the bar modelled as an one-dimensional continuum (forces acting on unit bar length). m_y and m_z are intensities of bending moments from volume forces about axes y and z, m_x is intensity of the torque due to volume forces and referred to the torsion axis of the bar, while $m_{\omega_s}(x, t)$ is intensity of the warping moment due to volume forces (moments acting on unit bar length).

The term for the work of surface forces will be expanded by means of (2.12.a—c) and (2.13.a—b).

$$\Phi(\mathbf{u})_d = \int_0^{t_0}\int_{A_d} \hat{t}(x, y, z, t)\,\mathbf{u}(x, y, z, t_0 - t)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t$$

$$= \int_0^{t_0}\int_A [\hat{\mathbf{t}}(0, y, z, t)\,\mathbf{u}(0, y, z, t_0 - t) +$$

$$+ \hat{\mathbf{t}}(l, y, z, t)\,\mathbf{u}(l, y, z, t_0 - t)]\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t$$

$$= \int_0^{t_0} \{- \int_A \widehat{\sigma}(0, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{xs}(0, t_0 - t) + \int_A \widehat{\sigma}(0, y, z, t)y\,\mathrm{d}y\,\mathrm{d}z\,u'_{ys}(0, t_0 - t) +$$

$$+ \int_A \widehat{\sigma}(0, y, z, t)z\,\mathrm{d}y\,\mathrm{d}z\,u'_{zs}(0, t_0 - t) + \int_A \widehat{\sigma}(0, y, z, t)\,\omega_s(y, z)\,\mathrm{d}y\,\mathrm{d}z\,\varphi'(0, t_0 - t) +$$

$$+ \int_A \widehat{\tau}_{xy}(0, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{ys}(0, t_0 - t) + \int_A \widehat{\tau}_{xz}(0, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{zs}(0, t_0 - t) -$$

$$- \left(\widehat{\tau}_{xz}(0, y, z, t)y - \widehat{\tau}_{xy}(0, y, z, t)z\right)\mathrm{d}y\,\mathrm{d}z\,\varphi(0, t_0 - t) +$$

$$+ \int_A \widehat{\sigma}(l, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{xs}(l, t_0 - t) - \int_A \widehat{\sigma}(l, y, z, t, t)y\,\mathrm{d}y\,\mathrm{d}z\,u'_{ys}(l, t_0 - t) -$$

$$- \int_A \widehat{\sigma}(l, y, z, t)z\,\mathrm{d}y\,\mathrm{d}z\,u'_{zs}(l, t_0 - t) -$$

$$
\begin{aligned}
&- \int_A \widehat{\sigma}(l, y, z, t)\,\omega_s(y, z)\,\mathrm{d}y\,\mathrm{d}z\,\varphi'(l, t_0 - t) - \\
&- \int_A \widehat{\tau}_{xy}(l, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{ys}(l, t_0 - t) - \int_A \widehat{\tau}_{xz}(l, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\,u_{zs}(l, t_0 - t) + \\
&+ \int_A \left(\widehat{\tau}_{xz}(l, y, z, t)y - \widehat{\tau}_{xy}(l, y, z, t)\,\mathrm{d}y\,\mathrm{d}z\right)\varphi(l, t_0 - t)\,\mathrm{d}t = \\
&= \int_0^{t_0} \{[\widehat{N}(x, t)\,u_{xs}(x, t_0 - t)]_{x=0}^{x=l} - \\
&- [\widehat{Q}_{ys}(y, t)\,u_{ys}(x, t_0 - t)]_{x=0}^{x=l} - [\widehat{M}_z(x, t)\,u'_{ys}(x, t_0 - t)]_{x=0}^{x=l} - \\
&- [\widehat{Q}_{zs}(x, t)\,u_{zs}(x, t_0 - t)]_{x=0}^{x=l} + [\widehat{M}_y(x, t)\,u'_{zs}(x, t_0 - t)]_{x=0}^{x=l} + \\
&+ [\widehat{M}_x(x, t)(x, t_0 - t)]_{x=0}^{x=l} - [B_s(x, t)\,\varphi'(x, t_0 - t)]_{x=0}^{x=l}\}
\end{aligned}
\tag{3.4}
$$

Here A_d means the surfaces of cross sections at points $x = 0$ and $x = l$.

Simplified symbols introduced for integrals on surface A are interpreted in Fig. 3 and the relevant comments.

Bending moments $\widehat{M}_y(0, t)$ and $\widehat{M}_y(l, t)$ are affected by negative sign since assumed displacement (2.) involves a bending moment pointing to the negative direction of the y-axis.

The term for the initial condition specified for velocity distribution is expanded by means of (2.12.a—c) and (2.14).

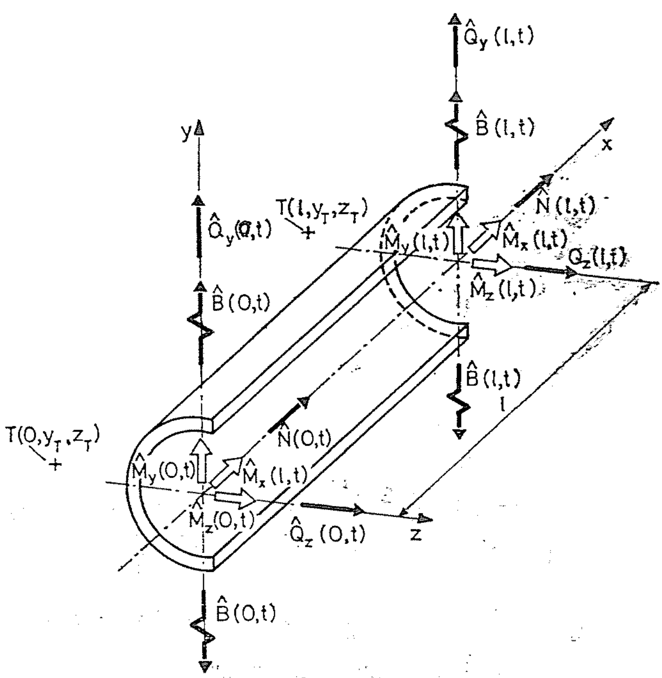

Fig. 3

$$\Phi(\mathbf{u})_v = \int_V \varrho(\mathbf{x})\mathbf{v}_0(\mathbf{x})\mathbf{u}(x, t_0 - t)\,\mathrm{d}V =$$

$$= \int_0^l \int_A \{\varrho[v_{xs}^0(x) - v_{ys}^{'0}(x)y - v_{zs}^{'0}(x)z - \varkappa'^0(x)\omega_s(y, z)] \cdot$$

$$\cdot [u_{xs}(x, t_0) - u'_{ys}(x, t_0)y - u'_{zs}(x, t_0)z - \varphi'(x, t_0)\omega_s(y, z)] +$$

$$+ [v_{ys}^0(x) - z\varkappa^0(x)][u_{ys}(x, t_0) - z\varphi(x, t_0)] +$$

$$+ [v_{zs}(x) + y\varkappa^0(x)][u_{zs}(x, t_0) + y\varphi(x, t_0)]\}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x =$$

$$= \int_0^l \{\varrho A\, v_{xs}^0(x)\, u_{xs}(x, t_0) +$$

$$+ \varrho A\, v_{ys}^0(x)\, u_{ys}(x, t_0) + \varrho I_{zz}\big(v_{ys}^{'0}(x) - z_T\varkappa^{'0}(x)\big)u'_{ys}(x, t_0) +$$

$$+ \varrho A\, v_{zs}^0(x)\, u_{zs}(x, t_0) + \varrho I_{yy}\big(v_{zs}^{'0}(x) + y_T\varkappa^{'0}(x)\big)u'_{zs}(x, t_0) +$$

$$+ \varrho I_{ps}\varkappa^0(x)\varphi(x, t_0) + \varrho[-I_{zz}z_T v_{ys}^{'0}(x) + I_{yy}y_T v_{zs}^{'0}(x) +$$

$$+ I_{\omega_s}\varkappa^{'0}(x)]\,\varphi'(x, t_0)\}\,\mathrm{d}x \tag{3.5}$$

Before writing functional $\Phi(u)$ in concrete form, let the following simplified notations be introduced:

$$\langle g, h\rangle_R = \int_0^{t_0}\int_0^l g(x, t)h(x, t_0 - t)\,\mathrm{d}x\,\mathrm{d}t \tag{3.6.a}$$

$$\langle g, h\rangle_{A_d} = \int_0^{t_0} [g(x, t)h(x, t_0 - t)]_{x=0}^{x=l}\,\mathrm{d}t \tag{3.6.b}$$

$$\langle g, h\rangle_0 = \int_0^l g(x, 0)h(x, t_0)\,\mathrm{d}x \tag{3.6.c}$$

Utilizing (1.3), (3.1) to (3.5) and (3.6.a):

$$\Phi(\mathbf{u}) = \Phi(\mathbf{u})_m + \Phi(\mathbf{u})_\varepsilon + \Phi(\mathbf{u})_f + \Phi(\mathbf{u})_d + \Phi(\mathbf{u})_{v_0} =$$

$$= \frac{1}{2}\varrho A\langle \dot{u}_{xs}, \dot{u}_{xs}\rangle_R + \frac{1}{2}EA\langle u'_{xs}, u'_{xs}\rangle_R - \langle q_x, u_{xs}\rangle_R -$$

$$- \langle \widehat{N}, u_{xs}\rangle_{A_d} - \varrho A\langle v_{xs}, u_{xs}\rangle_0 +$$

$$+ \frac{1}{2}\varrho A\langle \dot{u}_{ys}, \dot{u}_{ys}\rangle_R + \frac{1}{2}\varrho I_{zz}\langle \dot{u}'_{ys} - z_T\dot{\varphi}', u'_{ys}\rangle_R +$$

$$+ \frac{1}{2}EI_{zz}\langle u''_{ys} - z_T\varphi'', u''_{ys}\rangle_R -$$

$$- \langle q_y, u_{ys}\rangle_R - \langle m_z, u'_{ys}\rangle_R + \langle \widehat{Q}_{ys}, u_{ys}\rangle_{A_d} + \langle \widehat{M}_z, u'_{ys}\rangle_{A_d} -$$

$$- \varrho A\langle v_{ys}^0, u_{ys}\rangle_0 - \varrho I_{zz}\langle v_{ys}^{'0} - z_T\varkappa^{'0}, u'_{ys}\rangle_0 +$$

$$+ \frac{1}{2}\varrho A\langle \dot{u}_{zs}, \dot{u}_{zs}\rangle_R + \frac{1}{2}\varrho I_{yy}\langle \dot{u}'_{zs} + y_T\dot{\varphi}', \dot{u}'_{zs}\rangle_R +$$

$$+ \frac{1}{2}EI_{yy}\langle u''_{zs} + y_T\varphi'', u''_{zs}\rangle_R -$$

$$-\langle q_z, u_{zs}\rangle_R + \langle m_y, u'_{zs}\rangle_R + \langle \widehat{Q}_{zs}, u_{zs}\rangle_{A_d} - \langle \widehat{M}_y, u'_{zs}\rangle_{A_d} -$$
$$-\varrho A\langle v^0_{zs}, u_{zs}\rangle_0 - \varrho I_{yy}\langle v'^0_{zs} + y_T \varkappa'^0, u'_{zs}\rangle_0 +$$
$$+\frac{1}{2}\varrho I_{ps}\langle \dot\varphi, \dot\varphi\rangle_R + \frac{1}{2}\varrho\langle I_{\omega_s}\dot\varphi' - I_{zz}z_T\dot u'_{ys} + I_{yy}y_T\dot u'_{zs}, \dot\varphi'\rangle_R +$$
$$+\frac{1}{2}I_T G\langle \varphi', \varphi'\rangle_R + \frac{1}{2}E\langle I_{\omega_s}\varphi'' - I_{zz}z_T u''_{ys} + I_{yy}y_T u''_{zs}, \varphi\rangle_R -$$
$$-\langle m_x, \varphi\rangle_R + \langle m_{\omega_s}, \varphi'\rangle_R - \langle \widehat{M}_x, \varphi\rangle_{A_d} + \langle \widehat{B}_s, \varphi'\rangle_{A_d} -$$
$$-\varrho I_{ps}\langle \varkappa^0, \varphi\rangle_0 - \varrho\langle I_{\omega_s} - I_{zz}z_T v'^0_{ys} + I_{yy}y_T v'^0_{zs}, \varphi\rangle_0. \tag{3.7}$$

with coherent terms (scalar products) side by side. In Eq. (3.7) terms where the second factor is the same — irrespective of deriving with respect to place and time — belong together.

Displacement $\mathbf{u}(x, t)$ with the minimum of functional $\Phi(\mathbf{u})$ is known to meet also the motion equations wanted, that is, relationships for this displacement $u(x, t)$ yield the motion equations wanted.

To establish the equation for the minimum place of functional $\Phi(\mathbf{u})$,

$$\delta\Phi(\mathbf{u}, \delta\mathbf{u}) = 0 \tag{3.8}$$

has to be applied, where $\delta\Phi(\mathbf{u}, \delta\mathbf{u})$ is first variation of $\Phi(\mathbf{u})$ with respect to $\mathbf{u}$.

From (3, 7):

$$\delta\Phi(\mathbf{u}, \delta\mathbf{u}) = \langle \varrho A\dot u_{xs}, \varrho\dot u_{xs}\rangle_R + \langle EAu'_{xs}, \delta u'_{xs}\rangle_R - \langle q_x, \delta u_{xs}\rangle_R - \langle N, u_{xs}\rangle_{A_d} -$$
$$-\langle \varrho A v^0_{xs}, \delta u_{xs}\rangle_0 + \langle \varrho A\dot u_{ys}, \delta\dot u_{ys}\rangle_R + \langle \varrho I_{zz}(\dot u'_{ys} - z_T\dot\varphi'), \delta\dot u'_{ys}\rangle_R +$$
$$+\langle EI_{zz}(u''_{ys} - z_T\varphi''), \delta u''_{ys}\rangle_R -$$
$$-\langle q_y, \delta u_{ys}\rangle_R - \langle m_z, \delta u'_{ys}\rangle_R + \langle \widehat{Q}_{ys}, \delta u_{ys}\rangle_{A_p} + \langle \widehat{M}_z, \delta u'_{ys}\rangle_{A_d} -$$
$$-\langle \varrho A v^0_{ys}, \delta u_{ys}\rangle_0 - \langle \varrho I_{zz}(v'^0_{ys} - z_T\varkappa^0, \delta u'_{ys}\rangle_0 +$$
$$+\langle \varrho A\dot u_{zs}, \delta\dot u_{zs}\rangle_R + \langle \varrho I_{yy}(\dot u'_{zs} + y_T\dot\varphi'), \delta\dot u'_{zs}\rangle_R +$$
$$+\langle EI_{yy}(u''_{zs} + y_T\varphi'', \delta u''_{zs}\rangle_R -$$
$$-\langle q_z, \delta u_{zs}\rangle_R + \langle m_y, \delta u'_{zs}\rangle_R + \delta\langle \widehat{Q}_{zs}, \delta u_{zs}\rangle_R - \langle \widehat{M}_y, \delta u'_{zs}\rangle_{A_d} -$$
$$-\langle \varrho A v^0_{zs}, u_{zs}\rangle_0 - \langle \varrho I_{yy}(v'^0_{zs} + y_T\varkappa'^0), \delta u_{zs}\rangle_0 +$$
$$+\langle \varrho I_{ps}\dot\varphi, \delta\dot\varphi\rangle_R + \langle \varrho I_{\omega_s}\dot\varphi' - \varrho I_{zz}z_T\dot u'_{ys} + \varrho I_{yy}y_T\dot u'_{zs}, \delta\dot\varphi')\rangle_R +$$
$$+\langle GI_t\varphi', \delta\varphi'\rangle_R + \langle EI_{\omega_s}\varphi'' - EI_{zz}z_T u''_{ys} + EI_{yy}y_T u''_{zs}, \delta\varphi''\rangle_R -$$
$$-\langle m_x, \varrho\varphi\rangle_R + \langle m_{\omega_s}, \delta\varphi'\rangle_R - \langle \widehat{M}_x, \delta\varphi\rangle_{A_d} + \langle \widehat{B}_s, \delta\varphi'\rangle_{A_p} -$$
$$-\langle \varrho I_{ps}\varkappa^0, \delta\varphi\rangle_0 - \langle \varrho I_{\omega_s} - \varrho I_{zz}z_T v'^0_{ys} + \varrho I_{yy}y_T v'^0_{zs}, \delta\varphi'\rangle_0 \tag{3.9}$$

Possible reductions in (3.9) need transformation relationships

$$\langle g, h'\rangle_R = -\langle g', h\rangle_R + \langle g, h\rangle_{A_d}, \tag{3.10.a}$$
$$\langle g, \dot h\rangle_R = \langle \dot g, h\rangle_R + \langle g, h\rangle_0 - \langle h, g\rangle_0 \tag{3.10.b}$$

Validity of (3.10.a—b) is understood from defining equality (3.6.a), according to rules of partial integration with respect to place and time coordinates x and t, resp., and defining equalities (3.6.b—c).

Conveniently utilizing equalities (3.10.a—b) it is:

$$
\begin{aligned}
\delta\Phi(\mathbf{u}, \delta\mathbf{u}) = {} & \langle \varrho A \ddot{u}_{xs} - EA u''_{xs} - q_x, \delta u_{xs}\rangle_R + \langle EA u'_{xs} - \widehat{N}, \delta u_x \rangle_{A_d} + \\
& + \langle \varrho A \dot{u}_{xs} - \varrho A v^0_{xs}, \delta u_{xs}\rangle_0 + \\
& + \langle \varrho A \ddot{u}_{ys} - \varrho I_{zz}(\ddot{u}''_{ys} - z_T \ddot{\varphi}'') + EI_{zz}(u'^{V}_{ys} - z_T \varphi'^{V}) - q_y - m'_z, \delta u_{ys}\rangle_R + \\
& + \langle \varrho I_{zz}(\ddot{u}'_{ys} - z_T \ddot{\varphi}') - EI_{zz}(u'''_{ys} - z_T \varphi''') - m_z + \widehat{Q}_{ys}, \delta u_{ys}\rangle_{A_d} + \\
& + \langle EI_{zz}(u''_{ys} - z_T \varphi'') + \widehat{M}_z, \delta u'_{ys}\rangle_{A_d} + \langle \varrho A(u_{ys} - v_{ys}), \delta u_{ys}\rangle_0 + \\
& + \langle \varrho I_{zz}[(\dot{u}'_{ys} - z_T \dot{\varphi}') - (v'^0_{ys} - z_T \varkappa'^0), \delta u'_{ys}\rangle_0 + \\
& + \langle \varrho A \ddot{u}_{zs} - \varrho I_{yy}(\ddot{u}_{zs} + y_T \ddot{\varphi}'') + EI_{yy}(u'^{V}_{zs} + y_T \varphi'^{V}) - q_z - m'_y, \delta u_{ys}\rangle_R + \\
& + \langle \varrho I_{yy}(\ddot{u}'_{zs} + y_T \ddot{\varphi}') - EI_{yy}(u'''_{zs} + y_T \varphi''') + m_y + \widehat{Q}_{zs}, \delta u_{zs}\rangle_{A_d} + \\
& + \langle EI_{yy}(u''_{zs} + y_T \varphi'') - \widehat{M}_y, \delta u'_{zs}\rangle_{A_d} + \langle \varrho A(\dot{u}_{zs} - v^0_{zs}), \delta u_{zs}\rangle_0 + \\
& + \langle \varrho I_{yy}[(\dot{u}'_{zs} + y_T \dot{\varphi}') - (v'^0_{zs} + y_T \varkappa'^0)], \delta u'_{zs}\rangle_0 + \\
& + \langle \varrho I_{ps} \ddot{\varphi} - \varrho I_{\omega_s} \ddot{\varphi}'' + \varrho I_{zz} z_T \ddot{u}''_{ys} - \varrho I_{yy} y_T \ddot{u}''_{zs} + EI_{\omega_s} \varphi'^{V} - \\
& - EI_{zz} z_T u'^{V}_{ys} + EI_{yy} y_T u'^{V}_{zs} - I_t G\varphi'' - m_x - m'_\omega, \delta\varphi\rangle_R + \\
& + \langle \varrho I_{\omega_s} \ddot{\varphi}' - \varrho I_{zz} z_T \ddot{u}'_{ys} + \varrho I_{yy} y_T \ddot{u}'_{zs} + m_{\omega_s} - EI_{\omega_s} \varphi''' - EI_{zz} z_T u'''_{ys} + \\
& + EI_{yy} y_T u'''_{zs} + I_{pT} G\varphi' - \widehat{M}_x, \delta\varphi\rangle_{A_d} + \langle EI_{\omega_s} \varphi'' - EI_{zz} z_T u''_{ys} + \\
& + EI_{yy} y_T u''_{zs} + \widehat{B}_s, \delta\varphi'\rangle_{A_d} + \langle \varrho I_{ps}(\dot{\varphi} - \dot{\varkappa}), \delta\varphi\rangle_0 + \langle \varrho I_{\omega_s}(\dot{\varphi}' - \varkappa'^0) - \\
& - \varrho I_{zz} z_T (\dot{u}'_{ys} - v'^0_{ys}) + I_{yy} y_T (\dot{u}'_{zs} - v'^0_{zs}), \delta\varphi'\rangle_0
\end{aligned}
\tag{3.11}
$$

(Since kinematically possible variations of displacements and of anguar rotation $\delta u_x(x, t_0 - t)$, ... resp., (and their partial derivatives with respect to place coordinate x), meeting this restriction, may be arbitrary, making use of (3.11), (3.8) yields the wanted motion equations:

$$
\varrho A \ddot{u}_{xs}(x, t) - EA u''_{xs}(x, t) = q_x(x, t) \tag{3.12.a}
$$

$$
\begin{aligned}
& \varrho A \ddot{u}_{ys}(x, t) - \varrho I_{zz}\big(\ddot{u}''_{ys}(x, t) - z_T \ddot{\varphi}''(x, t)\big) + \\
& + EI_{zz}\big(u'^{V}_{ys}(x, t) - z_T \varphi'^{V}(x, t)\big) = q_y(x, t) - m'_z(x, t),
\end{aligned}
\tag{3.12.b}
$$

$$
\begin{aligned}
& \varrho A \ddot{u}_{zs}(x, t) - \varrho I_{yy}\big(\ddot{u}''_{zs}(x, t) + y_T \ddot{\varphi}''(x, t)\big) + EI_{yy}\big(u'^{V}_{zs}(x, t) + \\
& + y_T \varphi'^{V}(x, t)\big) = q_z(x, t) + m'_y(x, t)
\end{aligned}
\tag{3.12.c}
$$

$$
\begin{aligned}
& \varrho I_{ps} \ddot{\varphi}(x, t) - \varrho I_{\omega_s} \ddot{\varphi}''(x, t) + \varrho I_{zz} z_T \ddot{u}''_{ys}(x, t) - \varrho I_{yy} y_T \ddot{u}_{zs}(x, t) - \\
& - I_t G\varphi''(x, t) + EI_{\omega_s} \varphi'^{V}(x, t) - EI_{zz} z_T u'^{V}_{ys}(x, t) + \\
& + EI_{yy} y_T u'^{V}_{zs}(x, t) = m_x(x, t) + m'_{\omega_s}(x, t)
\end{aligned}
\tag{3.12.d}
$$

where $0 \leq x \leq l$ and $0 \leq t \leq t_0$: boundary conditions

$$EA\,u'_{xs}(x, t) - \widehat{N}(x, t) = 0 \tag{3.13.a}$$

$$\varrho I_{zz}[\ddot{u}'_{ys}(x, t) - z_T \ddot{\varphi}'(x, t) - EI_{zz}[u'''_{ys}(x, t) - z_T \varphi'''(x, t) - \\ - m_z(x, t) + \widehat{Q}_{ys}(x, t) = 0 \tag{3.13.b}$$

$$EI_{zz}[u''_{ys}(x, t) - z_T \varphi''(x, t)] + \widehat{M}_z(x, t) = 0 \tag{3.13.c}$$

$$\varrho I_{yy}[\ddot{u}_{zs}(x, t) + y_T \ddot{\varphi}(x, t)] - EI_{yy}[u'''_{zs}(x, t) + y_T \varphi'''(x, t)] + \\ + m_y(x, t) + \widehat{Q}_{zs}(x, t) = 0 \tag{3.13.d}$$

$$EI_{yy}\big(u''_{zs}(x, t) + y_T \varphi''(x, t)\big) - \widehat{M}_y(x, t) = 0 \tag{3.13.e}$$

$$\varrho I_{\omega_s} \ddot{\varphi}'(x, t) - \varrho I_{zz} z_T \ddot{u}'_{ys}(x, t) + \varrho I_{yy} y_T \ddot{u}'_{zs}(x, t) + \\ + m_{\omega_s}(x, t) - EI_{\omega_s} \varphi''(x, t) - EI_{zz} z_T u'''_{ys}(x, t) + EI_{yy} y_T u'''_{zs}(x, t) + \\ + I_t G \varphi'(x, t) - \widehat{M}_x(x, t) = 0 \tag{3.13.f}$$

$$EI_{\omega_s} \varphi''(x, t) - EI_{zz} z_T u''_{ys}(x, t) + EI_{yy} y_T u''_{zs}(x, t) + \widehat{B}_s(x, t) = 0 \tag{3.13.g}$$

where $x = 0$, or $x = l$, and $0 \leq t \leq t_0$, as well as initial conditions

$$\dot{u}_{xs}(x, 0) - v^0_{xs}(x) = 0 \tag{3.14.a}$$

$$\dot{u}_{ys}(x, 0) - v^0_{ys}(x) = 0 \tag{3.14.b}$$

$$\dot{u}'_{ys}(x, 0) - z_T \dot{\varphi}'(x, 0) - [v'^0_{ys}(x) - z_T \varkappa'^0(x)] = 0 \tag{3.14.c}$$

$$\dot{u}_{zs}(x, 0) - v^0_{zs}(x) = 0 \tag{3.14.d}$$

$$\dot{u}'_{zs}(x, 0) + y_T \dot{\varphi}'(x, 0) - [v'^0_{zs}(x) + y_T \varkappa'^0(x)] = 0 \tag{3.14.e}$$

$$\dot{\varphi}(x, 0) - \dot{\varkappa}(x) = 0 \tag{3.14.f}$$

$$I_{\omega_s}[\dot{\varphi}'(x, 0) - \varkappa'^0(x)] - I_{zz} z_T [\dot{u}'_{ys}(x, 0) - v'^0_{ys}(x)] + \\ + I_{yy} y_T [\dot{u}'_{zs}(x, 0) - v'^0_{zs}(x)] = 0 \tag{3.14.g}$$

where $0 \leq x \leq l$.

4. Conclusions

Motion equations of closed or solid section bars mostly are described by the coordinates of centre of gravity. Simultaneously modelling open and closed section bars in a framework it is expedient to apply the same type of co-ordinates. For this purpose the co-ordinates of centre of gravity are suitable.

For elaboration of equilibrium equations of thin walled bars in terms of coordinates of centre of gravity the principle of total potential energy is applied.

References

1. WITT, D.: Berüchsichtigung der Wölbtorsion bei der dinamischen stabtragwerksberechnung. Proc. 5. Tagung Festkörpermechanik, Dresden, Band B. VEB Fachbuchverlag, Leipzig, pp. XXV/1—XXV/10 (1982)
2. MICHELBERGER, P.—FEKETE, A.: Könnyűszerkezetek. Egyetemi jegyzet, Tankönyvkiadó, Budapest, 1982 (In Hungarian)
3. KUTI, I.: Application of the Principle of Total Potential Energy to Establish the Motion Equation of Thin-Walled Open Section Bars, Periodica Polytechnica (Transp. Eng.), 1987
4. GURTIN, M. E.: Variational Principles for Linear Initial Value Problems, Quart. Appl. Math. *22*, 252—256 (1964)
5. GURTIN, M. E.: Variational Principles for Linear Elastodynamics, Archive for Rational Mechanics and Analysis *16*, 34—50 (1964)
6. ODEN, J. T.—REDDY, J. N.: Variational methods in Theoretical Mechanics. Springer-Verlag, Berlin—Heidelberg—New York, 1976
7. REDDY, J. N.: A Note on Mixed Variational Principles for Initial Value Problems. Q. J. Mech. Appl. Math. *28*, 123—132 (1975)
8. REDDY, J. N.: Modified Gurtin's Variational Principles for Initial Value Problems in the Linear Dynamic Theory of Viscoelasticity. Int. J. Solids and Structures *12*, 227—235 (1976)
9. TONTI, E.: On the Variational Formulation for Linear Initial Value Problems. Annali di Matematica Pura ed Applicata *95*, 331—359 (1973)
10. KUTI, I.: On the Primal Variational Principles in Linear Elastodynamics, Acta Technica Academiae Scien. Hung., *93* (1—2), pp. 101—114 (1981)

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