APPLICATION OF THE PRINCIPLE OF TOTAL POTENTIAL ENERGY TO ESTABLISH THE MOTION EQUATION OF THIN-WALLED OPEN-SECTION BARS

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Summary

Linear elasto-dynamic variational principles explicitly comprising initial conditions have been developed in the '70s. The principle of total potential energy will be applied to establish the motion equation of thin-walled open-section bars.

Introduction

Open-section bars are common in skeletons and frameworks exposed to dynamic loads. (Skeletons and frameworks of buildings, technology equipment, vehicle undercarriages, etc.). These bars are mainly exposed to tension-compression, bending and torsion. Bar ends incorporated (usually welded) in skeletons or frameworks are not free to displace, thus, often, in addition to Saint-Venant torsion, also torsion due to inhibited warping has to be taken into consideration.

The motion (equilibrium) equation can be directly written on mechanical considerations, just as by using the total potential energy functional referring to the given single selected bar. This latter method has two noteworthy advantages. Partly, together with the motion equation, also boundary conditions are obtained so to say automatically (and also the initial conditions for the velocity) that are not simple even in this case. And partly, (approximate) solution of the motion equation may rely on common, efficient functional analytic methods, including the actually rather generalized finite element method.

Let us note that the discussed motion equation—without deduction and boundary conditions—is also found in [1].

1. The principle of total potential energy

Obviously, the so-called direct generalization of the scalar product (bilinear form) utilized for developing variational principles in linear elastostatics in the form

$$[U_1,\ U_2] = \int_0^t \int_V U_1(x,t)\,U_2(x,t)\ \mathrm{d}x\mathrm{d}t \tag{1.1}$$

is not symmetric about term $[\delta U/\delta t, U]$, therefore operators comprising operand $\frac{\partial \cdot}{\partial t}$ are not of the potential type, hence in general are unsuitable for handling initial conditions of linear elasto-dynamics. (In Eq. (1.1), V is a single coherent three-dimensional open domain, $0 \leq t \leq t_0$ a confined time interval, while $U_1(x, t)$, $U_2(x, t)$, $U(x, t)$ are quadratically integrable functions.)

Again, evidently, scalar product

$$\langle U_1, U_2 \rangle = \int_0^t \int_V U_1(x,t) \; U_2(x, t_0 - t) dx \; dt \tag{1.2}$$

is symmetric about term $\langle \delta U/\delta t, U \rangle$, that is, it perfectly suits the development of variational principles of linear elasto-dynamics.

Among relevant research, the most important ones are those due to Gurtin [2, 3], Tonti [4], Oden and Reddy [5] and Reddy [6, 7]. Gurtin was the first to apply scalar product (1.2) (convolution) for developing linear elasto-dynamic variational principles implicitly containing the initial conditions. Tonti demonstrated scalar product $\langle \delta U/\delta t, U \rangle$ to produce a symmetric variational principle referring to the thermal conduction equation.

Variational principles published by Oden and Reddy explicitly contained initial conditions.

In linear elasto-dynamics, like in elasto-statics, variational principles referring to the total potential energy, the complementary energy and the so-called Reissner variational principles are of practical importance. Actually, the principle of total potential energy will be involved, with the following so-called total energy functional:

$$\Phi(u) = \frac{1}{2}\int_0^{t_0}\int_V \varrho(x) \; \dot{u}(x,t)\dot{u}(x,t_0 - t) dx dt + \frac{1}{2}\int_0^{t_0}\int_V [\, \mathrm{E}(x) : \epsilon(x,t)]$$

$$: \epsilon(x,t_0 - t) \, dx dt - \int_0^{t_0}\int_V f(x,t) \; u(x,t_0 - t) \, dx dt$$

$$- \int_0^{t_0}\int_{A_d} \hat{t}(x,t) \; u(x,t_0 - t) dx dt - \int_V \varrho(x) \; v^0(x) u(x,t_0) \, dx, \tag{1.3}$$

where:

$0 \leq t \leq t$	confined time interval;
$x = x(x, y, z)$	coordinate of place of a point of the given solid;
V	domain occupied by the given solid;
A_p	boundary (surface) of domain V;
A_d	part of surface A_p with given surface forces;
$u = u(x, t)$	displacement vector field;
$\varrho = \varrho(x)$	volume intensity of mass distribution;
$\mathrm{E} = \mathrm{E}(x)$	fourth-order tensor of material characteristics;

$\varepsilon = \varepsilon(x, t)$ strain vector field;
$f = f(x, t)$ intensity of volume forces;
$\hat{t} = \hat{t}(x, t)$ intensity of prescribed surface forces on surface A_d;
$v^0 = v^0(x)$ initial velocity distribution specified for the given solid;
$(\dot{\,}) = \partial(.)/\partial t$ (symbol of partial derivation with respect to t)
$:$ symbol of twofold scalar multiplication.

Functional $\Phi(u)$ involves the following a priori conditions:

$$\sigma(x,t) = \mathrm{E}(x) : \epsilon(x,t) \tag{1.4}$$

$$\epsilon(x,t) = \frac{1}{2}\left[\nabla u(x,t) + (\nabla u(x,t))^T\right] \tag{1.5}$$

$$u(x,t) = \hat{u}(x,t), \quad x \in \mathrm{A}_u, \tag{1.6}$$

$$u(x,0) = u^0(x), \quad x \in \mathrm{V}, \tag{1.7}$$

where:

$\sigma = \sigma(x, t)$ stress tensor field;
$\nabla(.) =$ grad (.);
$\mathrm{A}_u =$ part of surface A_p where displacement is given as boundary condition, $A_p = A_d \cup A_u$, $A_d \cap A_u = \emptyset$ ($\emptyset$ is symbol of an empty set);
$\hat{u}(x, t) =$ specified displacement over surface A_u;
$u^0(x) =$ specified initial displacement over domain V.

(1.4) yields the material law, while (1.5) to (1.7) provide for the kinematic possibility of displacement $u(x, t)$. Deductions for functional $\Phi(u)$ and for condition (1.4) to (1.7) are found in [8].

2. Assumptions for writing the motion equation

The open-section bar has to be modelled as a one-dimensional continuum. The bar is assumed to be prismatic, slender, of a homogeneous, isotropic material. (Assumption of an orthotropic material causes no difficulties either.) Let the bar be exposed to external forces and moments seen in Fig. 1. and by volume force

$$f(x, y, z, t)^T = (f_x(x, y, z, t), f_y(x, y, z, t), f_z(x, y, z, t)). \tag{2.1}$$

In conformity with symbols in Fig. 1 $\hat{N}$ is the force along the bar, $\hat{Q}_y$, $\hat{Q}_z$ are shear forces, $\hat{M}_x$ the torque, $\hat{M}_y$, $\hat{M}_z$ are bending moments and $\hat{B}$ the so-called bimoment. Axes y and z are assumed to be principal axes of inertia of the bar cross section. $T(x, y_T, z_T)$ is the torsion center for the cross section of coordinate x.

Displacement of an arbitrary bar point is obtained from

$$u = (u_x(x,y,z,t),\ u_y(x,y,z,t),\ u_z(x,y,z,t)) \tag{2.2}$$

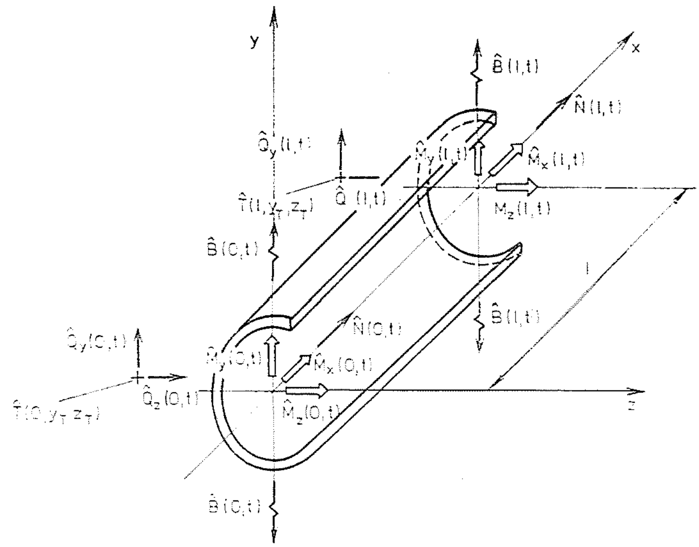

Fig. 1

$$u_x(x,y,z,t) = u_{xT}(x,t) - u'_{yT}(x,t)y - u'_{zT}(x,t)z -$$

$$- \varphi'(x,t)\omega(y,z), \tag{2.2.a}$$

$$u_y(x,y,z,t) = u_{yT}(x,t) - (z - z_T)\ \varphi(x,t), \tag{2.2.b}$$

$$u_z(x,y,z,t) = u_{zT}(x,t) + (y - y_T)\varphi(x,t), \tag{2.2.c}$$

where:

— $u_{xT}(x,t)$, $u_{yT}(x,t)$, $u_{zT}(x,t)$ displacement coordinates of an arbitrary point

— $(.)' = \partial(.)/\partial x$;

(x,t) of the (straight) torsion axis;

— $\varphi(x,t)$ angle of rotation of the cross section of coordinate x and normal to the $x - x$ axis about the torsion axis (positive if the vector of rotation points to the positive direction of the x-axis);

— $\omega_T(y,z)$ warping rate referred to torsion center T (determined clock-wise on a surface directed by an outer normal unit vector pointing to the negative direction of the x-axis).

Surface loads (stresses) specified for bar ends are described by equalities:

$$\hat{t}(O,y,z,t)^T = (-\hat{\sigma}(O,y,z,t),\ -\hat{\tau}_{xy}(O,y,z,t),\ -\hat{\tau}_{xz}(O,y,z,t)), \tag{2.3a}$$

$$\hat{t}(l,y,z,t)^T = (\hat{\sigma}(l,y,z,t),\ \hat{\tau}_{xy}(l,y,z,t),\ \hat{\tau}_{xz}(l,y,z,t)), \tag{2.3.b}$$

(Negative sign in (2.3.a) refers to the surface of an outer normal pointing to the negative direction.) Stresses $\hat{t}$ $(0, y, z, t)$ and $\hat{t}$ (l, y, z, t) are assumed to arise as sums of stresses corresponding to elementary ones acting on the bar.

Initial velocity distribution has to be specified according to the assumed distribution field u ((2.2) to (2.2.c)), that is:

$$v^0(x,y,z,t)^T = (v^0_{xT}(x) - v'^0_{yT}(x)y - v'^0_{zT}(x)z - \varkappa'^0(x)\omega_T(y,z),$$
$$v^0_{yT}(x) - (z - z)\varkappa^0(x), \quad v^0_{zT}(x) + (y - y_T)\varkappa^0(x)), \ (2.4)$$

where v^0_{xT} v^0_{yT}, v^0_{zT}, $\varkappa^0$, v'^0_{yT}, v'^0_{zT} and $\varkappa'^0$ are initial values at time $t = 0$ of velocities and angular velocities $\dot{u}_{xT}$, $\dot{u}_{yT}$, $\dot{u}_{zT}$, $\dot{\varrho}$, $\dot{u}'_{yT}$, $\dot{u}'_{zT}$ and/or of their derivatives with respect to x.

Obviously, also kinematic boundary condition $\hat{u}(x.t)$ and initial displacement $u^0(x)$ in conditions (1.6) and (1.7) have to be specified in conformity with the assumed displacement field (2.2) to (2.2.c).

Deformation of open-section bars have been detailed in [9] and [10].

3. Establishment of the motion equation relying on the principle of total potential energy

For the sake of understanding, functional $\Phi(u)$ will be written in the concrete form for the examined problem term-wise, and after simplifying notations, each term will be summed in conformity with (1.3).

Expanding term for the kinetic energy by means of Eqs (2.2) to (2.2.c):

$$\Phi(u)_m = \frac{1}{2}\int_0^{t_0}\int_v \varrho(x)\dot{u}(x,t)\,\dot{u}(x,t_0 - t)\ \mathrm{d}x\ dt$$

$$= \frac{1}{2}\int_0^{t_0}\int_0^l\int_A \varrho[\dot{u}_{xT}(x,t) - \dot{u}'_{yT}(x,t)\,y - \dot{u}'_{zT}(x,t)\ z - \dot{\varphi}'(x,t)\ \omega_T(y,z)]$$
$$[\dot{u}_{xT}(x,t_0 - t) - \dot{u}'_{yT}(x,t_0 - t)\,y - \dot{u}'_{zT}(x,t_0 - t)z - \dot{\varphi}'(x,t_0 - t)\ \omega_T(y,z)]$$
$$\cdot\ dy\ dz\ dx\ dt$$
$$+ \frac{1}{2}\int_0^{t_0}\int_0^l\int_A \varrho[\dot{u}_{yT}(x,t) - (z - z_T)\dot{\varphi}(x,t)][\dot{u}_{yT}(x,t_0 - t) - (z - z_T)\dot{\varphi}(x,t_0 - t)]$$
$$\cdot\ dy\ dz\ dx\ dt$$
$$+ \frac{1}{2}\int_0^{t_0}\int_0^l\int_A \varrho[\dot{u}_{zT}(x,t) + (y - y_T)\dot{\varphi}(x,t)][\dot{u}_{zT}(x,t_0 - t) + (y - y_T)\dot{\varphi}(x,t_0 - t)]$$
$$\cdot\ dy\ dz\ dx\ dt$$
$$= \frac{1}{2}\varrho A\int_0^{t_0}\int_0^l \dot{u}_{xT}(x,t)\ \dot{u}_{xT}(x,t_0 - t)\ dx\ dt$$

$$+\frac{1}{2}\varrho A\int_0^{t_0}\int_0^l (\dot{u}_{yT}(x,t)+z_T\dot{\varphi}(x,t)]\dot{u}_{yT}(x,t_0-t)\ dx\ dt$$

$$+\frac{1}{2}\varrho I_{zz}\int_0^{t_0}\int_0^l \dot{u}'_{yT}(x,t)\dot{u}'_{yT}(x,t_0-t)\ dx\ dt$$

$$+\frac{1}{2}\varrho A\int_0^{t_0}\int_0^l [\dot{u}_{zT}(x,t)-y_T\dot{\varphi}(x,t)]\dot{u}_{zT}(x,t_0-t)dx\ dt$$

$$+\frac{1}{2}\varrho I_{yy}\int_0^{t_0}\int_0^l [\dot{u}'_{zT}(x,t)\dot{u}'_{zT}(x,t_0-t)\ dx\ dt$$

$$+\frac{1}{2}\varrho\int_0^{t_0}\int_0^l [I_{pT}\dot{\varphi}(x,t)-A(y_T\dot{u}_{zT}(x,t)-z_T\dot{u}_{yT}(x,t)]\ \dot{\varphi}(x,t_0-t)\ dx\ dt$$

$$+\frac{1}{2}\varrho I\omega\int_0^{t_0}\int_0^l \dot{\varphi}'(x,t)\ \dot{\varphi}'(x,t_0-t)\ dx\ dt, \tag{2.5}$$

where:

ϱ = constant, mass distribution intensity;

A = bar cross section area;

I_{yy} and I_{zz} second-order moments of inertia about axes y and z of the bar cross section;

I_{pT} (polar) second-order moment of inertia of the bar cross section referring to torsion center T;

$I_\omega = \int_A \omega_T^2(y,z)\ dy\ dz$ second-order moment of warping.

In calculating integrals with respect to surface A, it has been taken into consideration that

$$\int_A y\ dy\ dz = 0, \int_A z\ dy\ dz = 0, \int_A yz\ dy\ dz = 0,$$

and assumed that in determining the distortion rate $\omega_T(y,z)$ the origin is chosen to meet relationships

$$\int_A \omega_T(y,z)dy\ dz = 0, \quad \int_A y\omega_T(y,z)dy\ dz = 0,$$

$$\int_A z\omega_T(y,z)\ dy\ dz = 0.$$

The term for strain energy is expanded to:

$$\Phi(u)_\varepsilon = \frac{1}{2}\int_0^{t_0}\int_v [\mathrm{E}(x) : \epsilon(x,t)] : \epsilon(x,t_0 - t)\mathrm{d}x\, dt$$

$$= \frac{1}{2}\mathrm{E}\int_0^{t_0}\int_0^{l}\int_A \epsilon_x(x,y,z,t) \cdot \epsilon_x(x,y,z,t_0 - t)\; dy\; dz\; dx\; dt$$

$$+ \frac{1}{2}I_tG \int_0^{t_0}\int_0^{l} \varphi'(x,t)\; \varphi'(x,t_0 - t)dx\; dt$$

$$= \frac{1}{2}E\int_0^{t_0}\int_0^{l}\int_A [u'_{xT}(x,t) - u''_{yT}(x,t)y - u''_{zT}(x,t)z - \varphi''(x,t)\omega_T(y,z)]$$

$$\cdot\; [u'_{xT}(x,t_0 - t) - u''_{yT}(x,t_0 - t)y - u''_{zT}(x,t_0 - t)z - \varphi''(x,t_0 - t)\omega_T(y,z)]$$

$$\cdot\; dy\; dz\; dx\; dt$$

$$+ \frac{1}{2}I_tG \int_0^{t_0}\int_0^{l} \varphi'(x,t)\; \varphi'(x,t_0 - t)dx\; dt$$

$$= \frac{1}{2}E\,A \int_0^{t_0}\int_0^{l} u'_{xT}(x,t)\;\; u'_{xT}(x,t_0 - t)\; dx\; dt$$

$$+ \frac{1}{2}EI_{zz} \int_0^{t_0}\int_0^{l} u''_{yT}(x,t)\;\; u''_{yT}(x,t_0 - t)\;\; dx\; dt$$

$$+ \frac{1}{2}E\,I_{yy} \int_0^{t_0}\int_0^{l} u''_{zT}(x,t)\;\; u''_{zT}(x,t_0 - t)\;\; dx\; dt$$

$$+ \frac{1}{2}EI_\omega \int_0^{t_0}\int_0^{l} \varphi''(x,t)\;\; \varphi''(x,t_0 - t)\; dx\; dt\; +$$

$$+ \frac{1}{2}I_tG \int^{t_0}\int^{l} \varphi'(x,t)\; \varphi'(x,t_0 - t)\; dx\; dt \tag{2.6}$$

where;

$E = \dfrac{E^*}{1 - \nu^2}$, $G = \dfrac{E^*}{2(1 + \nu)}$; E^* is Young's modulus; and ν Poisson's ratio;

I_t second-order moment of Saint Venant torsion of the bar cross section;

$$\varepsilon_x(x,y,z,t) = \frac{\partial u_x(x,y,z,t)}{\partial x} .$$

The term for volume force work is expanded by means of (2.1) to (2.2.c).

$$\begin{aligned}
\Phi(u)_f &= \int_0^{t_0} \int_V f(x,t)\; u(x,t_0 - t)\; dx\, dt \\
&= \int_0^{t_0} \int_l \Big\{ \int_A f_x(x,y,z,t)\; dy\; dz\; u_{xT}(x,t_0 - t) \\
&- \int_A f_x(x,y,z,t)\, y\; dy\; dz\; u'_{yT}(x,t_0 - t) \\
&- \int_A f_x(x,y,z,t) z\; dy\; dz\; u'_{zT}(x,t_0 - t) - \int_A f_x(x,y,z,t)\; \omega_T(y,z) \\
&\cdot dy\; dz\; \varphi'(x,t_0 - t) \\
&+ \int_A f_y(x,y,z,t) dy\; dz\; u_{yT}(x,t_0 - t) + \int_A f_z(x,y,z,t)\; dy\; dz \\
&\cdot u_{zT}(x,t_0 - t) \\
&+ \int_A [(y - y_T) f_z(x,y,t,z) - (z - z_T) f_y(x,y,z,t)]\; dy\; dz \\
&\cdot \varphi(x,t_0 - t)\Big\}\; dx\; dt \\
&= \int_0^{t_0} \int_0^{l} \{q_x(x,t)\; u_{xT}(x,t_0 - t) \\
&+ q_y(x,t)\; u_{yT}(x,t_0 - t) + m_z(x,t)\; u'_{yT}(x,t_0 - t) \\
&+ q_z(x,t)\; u_{zT}(x,t_0 - t) - m_y(x,t) u'_{zT}(x,t_0 - t) \\
&+ m_{xT}(x,t)\varphi(x,t_0 - t) - m_\omega(x,t)\varphi'(x,t_0 - t)\} dx\; dt, \qquad (2.7)
\end{aligned}$$

introducing simplifying notations for integrals on surface A, with the following meanings:

q_x, q_y, q_z are intensities in directions x, y and z of volume forces acting on the bar modelled as a one-dimensional continuum (forces acting on unit bar length). m_y and m_z are intensities of bending moments from volume forces about axes y and z, m_{xT} is intensity of the torque due to volume forces and referred to the torsion axis of the bar, while $m_\omega(x,t)$ is the intensity of the warping moment due to volume forces (moments acting on unit bar length).

The term for the work of surface forces will be expanded by means of (2.2) to (2.3.b).

$$\begin{aligned}
\Phi(u)_d &= \int_0^{t_0} \int_{A_d} t(x,t)\; u(x,t_0 - t)\; dx\; dt \\
&= \int_0^{t_0} \int_A [\hat{t}(O,y,z,t)\; u(O,y,z,t_0 - t) + \hat{t}(l,y,z,t)\; u(l,y,z,t_0 - t)
\end{aligned}$$

$$
\begin{aligned}
&\cdot\, dy\, dz\, dt \\
&= \int_0^{t_0} \{ -\int_A \hat{\sigma}(O,y,z,t)\, dy\, dz\;\; u_{xT}(O,t_0 - t) + \\
&+ \int_A \hat{\sigma}(O,y,z,t)\; y\, dy\, dz\;\; u'_{yT}(O,t_0 - t) \\
&+ \int_A \hat{\sigma}(O,y,z,t)\; z\, dy\, dz\; u'_{zT}(O,t_0 - t) + \\
&+ \int_A \hat{\sigma}(O,y,z,t)\; \omega_T(y,z)\;\; dy\, dz \varphi'(O,t_0 - t) \\
&+ \int_A \hat{\tau}_{xy}(O,y,z,t)\, dy\, dz\;\; u_{yT}(O,t_0 - t) \\
&+ \int_A \hat{\tau}_{xz}(O,y,z,t)\;\; dy\, dz\;\; u_{zT}(O,t_0 - t) \\
&- \int_A [\hat{\tau}_{xz}(O,y,z,t)\;\; (y - y_T) - \hat{\tau}_{xy}(O,y,z,t)(z - z_T)]\, dy\, dz\; \varphi(O,t_0 - t) \\
&+ \int_A \hat{\sigma}(l,y,z,t)\, dy\, dz\;\; u_{xT}(l,t_0 - t) \\
&- \int_A \hat{\sigma}(l,y,z,t) y\; dy\, dz\;\; u'_{yT}(l,t_0 - t) \\
&- \int_A \hat{\sigma}(l,y,z,t)\; z\; dy\, dz\;\; u'_{zT}(l,t_0 - t) \\
&- \int_A \hat{\sigma}(l,y,z,t)\; \omega_T(y,z)\;\; dy\, dz\;\; \varphi'(l,t_0 - t) \\
&- \int_A \hat{\tau}_{xy}(l,y,z,t) dy\, dz\;\; u_{yT}(l,t_0 - t) - \int_A \hat{\tau}_{xz}(l,y,z,t) dy\, dz\;\; u_{zT}(l,t_0 - t) \\
&+ \int_A [\hat{\tau}_{xz}(l,y,z,t)\;\; (y - y_T) - \hat{\tau}_{xy}(l,y,z,t)(z - z_T)]\, dy\, dz \varphi(l,t_0 - t)\}\;\; dt
\end{aligned}
$$

$$
\begin{aligned}
= \int_A^{t_0} &\{ [\widehat{N}(x,t)\;\; u_{xT}(x,t_0 - t)]_{x=0}^{x=l} \\
&- [\widehat{Q}_y(x,t)\;\; u_{yT}(x,t_0 - t)]_{x=0}^{x=l} - [\widehat{M}_z(x,t)\;\; u'_{yT}(x,t_0 - t)]_{x=0}^{x=l} \\
&- [\widehat{Q}_z(x,t)\;\; u_{zT}(x,t_0 - t)]_{x=0}^{x=l} + [\widehat{M}_y(x,t)\;\; u'_{zT}(x,t_0 - t)]_{x=0}^{x=l} \\
&+ [\widehat{M}_x(x,t)\; \varphi(x,t_0 - t)]_{x=0}^{x=l} - [\widehat{B}(x,t)\; \varphi'(x,t_0 - t)]_{x=0}^{x=l}\}\;\; dt\,.
\end{aligned}
\tag{2.8}
$$

Simplified symbols introduced for integrals on surface A are interpreted in Fig. 1 and the relevant comments.

Bending moments $M_y\,(0,t)$ and $M_y(l,t)$ are affected by the negative sign since assumed displacement (2.2) involves a bending moment pointing to the negative direction of the y-axis.

The term for the initial condition specified for velocity distribution is expanded by means of (2.2) to (2.2.c) and (2.4).

$$
\Phi(u)_{v_0} = \int_V \varphi(x)\;\; v_0(x)\;\; u(x,t_0 - t)\; \mathrm{d}x
$$

$$= \int_0^{l_0} \varrho \int_A \{[v^0_{xT}(x) - v'^0_{yT}(x)y - v'^0_{zT}(x)z - \varkappa'^0(x)\,\omega_T(y,z)]$$
$$\cdot [u_{xT}(x,t_0 - u'_{yT}(x,t_0)\,y - u'_{zT}(x,t_0)z - \varphi'(x,t_0)\,\omega(y,z)]$$
$$+ [v_{yT}(x) - (z - z_T)\varkappa^0(x)]\ [u_{yT}(x,t_0) - (z - z_T)\ \varphi(x,t_0)]$$
$$+ [v^0_{zT}(x) + (y - y_T)\varkappa^0(x)][u_{zT}(x,t_0) + (y - y_T)\varphi(x,t_0)]\}\ dy\ dz\ dx$$
$$= \int_0^l \{\varrho A v^0_{xT}(x)\ u_{xT}(x, t_0)$$
$$+ \varrho A(v^0_{yT}(x) + z_T\varkappa^0(x))\ u_{yT}(x,t_0) + \varrho I_{zz} v'^0_{yT}(x)\ u'_{yT}(x,t_0)$$
$$+ \varrho A(v^0_{zT}(x) + z\varkappa^0(x))\ u_{zT}(x,t_0) + \varrho I_{yy} v'^0_{zT}(x)\ u'_{zT}(x,t_0)$$
$$+ [-\varrho A(v^0_{zT}(x)y_T - v'_{yT}(x)z_T) + \varrho I_{pT}\varkappa^0(x)]\ \varphi(x,t_0)$$
$$+ \varrho I_\omega \varkappa'^0(x)\ \varphi'(x,t_0)\}\ dx. \tag{2.9}$$

Before writing functional $\Phi(u)$ in concrete form, let the following simplified notations be introduced:

$$\langle g,h\rangle_R = \int_0^{t_0}\int_0^l g(x,t)\ h(x,t_0 - t)\ dx\ dt \tag{2.10.a}$$

$$\langle g,h\rangle_{A_d} = \int_0^{t_0} [g(x,t)\ h(x,t_0 - t)]_{x=0}^{x=l}\ dt \tag{2.10.b}$$

$$\langle g,h\rangle_0 = \int_0^l g(x,o)\ h(x,t_0)\ dx\,. \tag{2.10.c}$$

Utilizing (1.3), (2.5) to (2.9) and (2.10.a—c):

$$\Phi(u) = \Phi(u)_m + \Phi(u)_\varepsilon - \Phi(u)_f - \Phi(u)_d - \Phi(u)_{v^0}$$
$$= \frac{1}{2}\varrho A\,\langle \dot u_{xT}, \dot u_{xT}\rangle_R + \frac{1}{2}E\,A\langle u'_{xT}, u'_{xT}\rangle_R$$
$$- \langle \widehat N, u_{xT}\rangle_{A_d} - \varphi A\langle v^0_{xT}, u_{xT}\rangle_0$$
$$+ \frac{1}{2}\varrho A\langle \dot u_{yT} + z_T\dot\varphi,\ \dot u_{yT}\rangle_R +$$
$$+ \frac{1}{2}\varrho I_{zz}\langle \dot u'_{yT}, \dot u'_{yT}\rangle_R + \frac{1}{2}EI_{zz}\langle u''_{yT}, u''_{yT}\rangle_R$$
$$- \langle q_y, u_{yT}\rangle_R - \langle m_z, u'_{yT}\rangle_R + \langle \widehat Q_y, u_{yT}\rangle_{A_d} + \langle \widehat M_z, u'_{yT}\rangle_{A_d}$$
$$- \varrho A\langle v^0_{yT} + z_T\varkappa^0, u_{yT}\rangle_0 - \varrho I_{zz}\langle v'^0_{yT}, u'_{yT}\rangle_0$$
$$+ \frac{1}{2}\varrho A\langle \dot u_{zT} - y_T\dot\varphi, \dot u_{zT}\rangle_R +$$
$$+ \frac{1}{2}\varrho I_{yy}\langle \dot u'_{zT}, \dot u'_{zT}\rangle_R + \frac{1}{2}EI_{yy}\langle u''_{zT}, u''_{zT}\rangle_R$$

$$
\begin{aligned}
&- \langle q_z, u_{zT}\rangle_R + \langle m_y, u'_{zT}\rangle_R + \langle \widehat{Q}_z, u_{zT}\rangle_{A_d} - \langle \widehat{M}_y, u'_{zT}\rangle_{A_d} \\
&- \varrho A\langle v^0_{zT} - y_T\varkappa^0,\ u_{zT}\rangle_0 - \varrho I_{yy}\langle v'^0_{zT}, u_{zT}\rangle_0 \\
&+ \frac{1}{2}\varrho\langle I_{pT}\dot{\varphi} - A(\dot{u}_{zT}y_T - \dot{u}_{yT}Z_T),\ \dot{\varphi}\rangle_R + \\
&+ \frac{1}{2}\varrho I_\omega\langle\dot{\varphi}', \dot{\varphi}'\rangle_R + \frac{1}{2}EI_\omega\langle\varphi'', \varphi''\rangle_R \\
&+ \frac{1}{2}I_tG\langle\varphi', \varphi'\rangle_R - \langle m_{xT}, \varphi\rangle_R + \langle m_\omega, \varphi'\rangle_R - \langle M_x, \varphi\rangle_{A_d} + \\
&+ \langle \widehat{B},\widehat{\varphi}\rangle_{A_d} - \varrho\langle -A(v^0_{zT}y_T - v^0_{yT}z_T) + I_{pT}\varkappa^0,\varphi\rangle_0 - \varrho I_\omega\langle\varkappa'^0,\varphi'\rangle_0,
\end{aligned}
\tag{2.11}
$$

with coherent terms (scalar products) side by side. In Eq. (2.11), terms where the second factor is the same—irrespective of deriving with respect to place and time—belong together.

Displacement $u(x,t)$ with the minimum of functional 0 u is known to meet also the motion equations wanted, that is, relationships for this displacement $u(x,t)$ yield the motion equations wanted.

To establish the equation for the minimum place of functional $\Phi(u)$,

$$\delta\Phi(u,\ \delta u) = 0 \tag{2.12}$$

has to be applied, where $\delta\Phi(u, \delta u)$ is first variation of $\Phi(u)$ with respect to u. From (2.11):

$$
\begin{aligned}
&\delta\Phi(u,\ \delta u) \\
&= \langle \varrho A\dot{u}_{xT}, \delta\dot{u}_{xT}\rangle_R + \langle EAu'_{xT},\delta u'_{xT}\rangle_R - \langle q_x, \delta u_{xT}\rangle_R \\
&- \langle \widehat{N}, \delta u_{xT}\rangle_{A_d} - \langle \varrho Av^0_{xT}, \delta u_{xT}\rangle_0 \\
&+ \langle \varrho A(\dot{u}_{yT} + z_T\dot{\varphi},\ \delta\dot{u}_{yT}\rangle_R + \langle \varrho I_{zz}\dot{u}'_{yT}, \delta\dot{u}'_{yT}\rangle_R \\
&+ \langle EI_{zz}u''_{yT},\ \delta u''_{yT}\rangle_R - \langle q_y, \delta_{yT}\rangle_R \\
&- \langle m_z, \delta u'_{yT}\rangle_R + \langle \widehat{Q}_y, \delta u_{yT}\rangle_{A_d} + \langle \widehat{M}_z, \delta u'_{yT}\rangle_{A_d} \\
&- \langle \varrho A(v^0_{yT} + z_T\varkappa^0),\ \delta u_{yT}\rangle_0 - \langle \varrho I_{zz}v'^0_{yT},\delta u'_{yT}\rangle_0 \\
&+ \langle \varrho A(\dot{u}_{zT} - y_T\dot{\varphi}),\ \delta\dot{u}_{zT}\rangle_R + \langle \varrho I_{yy}\dot{u}'_{zT},\ \delta\dot{u}'_{zT}\rangle_R \\
&+ \langle EI_{yy}u''_{zT},\delta u''_{zT}\rangle_R - q_z,\delta u_{zT}\rangle_R \\
&+ \langle m_y, \delta u'_{zT}\rangle_R + \langle \widehat{Q}_z,\delta u_{zT}\rangle_{A_d} - \langle \widehat{M}_y,\delta u'_{zT}\rangle_{A_d} \\
&- \langle \varrho A(v^0_{zT} - y_T\varkappa^0),\ \delta u_{zT}\rangle_0 - \langle \varrho I_{yy}v'^0_{zT},\ \delta u'_{zT}\rangle_0 \\
&+ \langle \varrho I_{pT}\dot{\varphi} - \varrho A(\dot{u}_{zT}y_T - \dot{u}_{yT}z_T),\ \delta\dot{\varphi}\rangle_R \\
&+ \langle \varrho I_\omega\dot{\varphi}',\delta\dot{\varphi}'\rangle_R + \langle EI_\omega\varphi'', \delta\varphi''\rangle_R \\
&+ \langle I_tG\varphi',\ \delta\varphi'\rangle - \langle m_{xT},\delta\varphi\rangle_R + \langle m_\omega,\delta\varphi'\rangle_R - \langle \widehat{M}_x\delta\varphi\rangle_{A_d}
\end{aligned}
$$

$$+ \langle \hat{B},\delta\varphi' \rangle_{A_d} - \langle -\varrho A(v^0_{zT}y_T - v^0_{zT}z_T)$$
$$+ \langle \varrho I_{pT}\varkappa^0,\delta\varphi \rangle_0 - \langle \varrho I_\omega \varkappa'^0,\delta\varphi' \rangle_0 . \tag{2.13}$$

Possible reductions in (2.13) need transformation relationships

$$\langle g, h' \rangle_R = -\langle g', h \rangle_R + \langle g, h \rangle_{A_d}, \tag{2.14.a}$$

and

$$\langle g, h \rangle_R = \langle \dot{g}, h \rangle_R + \langle g, h \rangle_0 - \langle h, g \rangle_0 . \tag{2.14.b}$$

Validity of (2.14.1—b) is understood from defining equality (2.10.a), according to rules of partial integration with respect to place and time coordinates x and t, resp., and defining equalities (2.10.b—c).
Conveniently utilizing equalities (2.14.a—b) it is:

$$\begin{aligned}
&\delta\Phi(u, \delta u)\\
&= \langle \varrho A\ddot{u}_{xT} - EAu''_{yT} - q_x,\delta u_{xT} \rangle_R + \langle EA\, u'_{xT} - \hat{N}_{xT},\delta u_{xT} \rangle_{A_d}\\
&+ \langle \varrho A\dot{u}_{xT} - \varrho A v^0_{xT},\delta u_{xT} \rangle_0\\
&+ \langle \varrho A(\ddot{u}_{yT} + z_T\ddot{\varphi}) - \varrho I_{zz}\ddot{u}''_{yT} + EI_{zz}u^{IV}_{yT} - q_y + m'_z,\delta u_{yT} \rangle_R\\
&+ \langle \varrho I_{zz}\ddot{u}'_{yT} - EI_{zz}u'''_{yT} - m_z + \hat{Q}_y,\delta u_{yT} \rangle_{A_d}\\
&+ \langle EI_{zz}u''_{yT} + \hat{M}_z,\delta u'_y \rangle_{A_d}\\
&+ \langle \varrho A[(\dot{u}_{yT} + z_T\dot{\varphi}) - (v^0_{yT} + z_T\varkappa^0), \delta u_{yT} \rangle_0\\
&+ \langle \varrho I_{zz}(\dot{u}'_{yT} - v'^0_{yT}), \delta u'_{yT} \rangle_0 + \langle \varrho A(\ddot{u}_{zT} - y_T\ddot{\varphi}) -\\
&- \varrho I_{yy}\ddot{u}''_{zT} + EI_{yy}u^{IV}_{zT} - q_z - m'_y, \delta u_{zT} \rangle_R\\
&+ \langle \varrho I_{yy}\ddot{u}'_{zT} - EI_{yy}u'''_{zT} + m_y + \hat{Q}_z, \delta u_{zT} \rangle_{A_d}\\
&+ \langle EI_{yy}u''_{zT} - \hat{M}_y, \delta u'_{zT} \rangle_{A_d}\\
&+ \langle \varrho A[(\dot{u}_{zT} - y_T\dot{\varphi}) - (\dot{v}_{zT} - y_T\varkappa^0,\delta u_{yT} \rangle_0\\
&+ \langle \varrho I_{yy}(\dot{u}'_{zT} - v'^0_{zT}, \delta u'_{zT} \rangle_0 + \langle \varrho I_{pT}\ddot{\varphi} - \varrho A(\ddot{u}_{zT}y_T - \ddot{u}_{yT}z_T)\\
&- \varrho I_\omega\ddot{\varphi}'' - I_tG\varphi'' + EI_\omega\varphi^{IV} - m_{xT} - m'_\omega, \delta\varphi \rangle_R\\
&+ \langle \varrho I_\omega\ddot{\varphi}' + m_\omega - EI_\omega\varphi''' + I_tG\varphi' - \hat{M}_x,\delta\varphi \rangle_{A_d}\\
&+ \langle EI_\omega\varphi'' + \hat{B},\delta\varphi' \rangle_{A_d} + \langle \varrho I_{pT}(\dot{\varphi} - \varkappa^0) + \varrho A[(v^0_{zT}y_T - v^0_{yT}z_T)\\
&- (\dot{u}_{zT}y_T - \dot{u}_{yT}z_T),\delta\varphi \rangle_0 + \langle \varrho I_\omega\dot{\varphi}' - \varrho I_\omega\varkappa'^0,\delta\varphi' \rangle_0 .
\end{aligned} \tag{2.15}$$

Since kinematically possible variations of displacements and of angular rotation $\delta u_{xT}(x,t_0-t)$, $\delta u_{yT}(x,t_0-t)$, $\delta u_{zT}(x,t_0-t)$, $\delta\varphi(x,t_0-t)$ resp., (and their partial derivatives with respect to place coordinate x), meeting this restriction, may be arbitrary, making use of (2.12), (2.15) yields, the wanted motion equations:

$$\varrho A\ddot{u}_{xT}(x,t) - EAu''_{xT}(x,t) \qquad\qquad = q_x(x,t), \tag{2.16.a}$$

$$\varrho A(\ddot{u}_{yT}(x,t) + z_T\ddot{\varphi}(x,t) - \varrho I_{zz}\ddot{u}''_{yT}(x,t) + EI_{zz}u^{\mathrm{IV}}_{yT}(x,t)$$
$$= q_y(x,t) - m'_z(x,t), \tag{2.16.b}$$
$$\varrho A(\ddot{u}_{zT}(x,t) - y_T\ddot{\varphi}(x,t) - \varrho I_{yy}\ddot{u}''_{zT}(x,t) + EI_{yy}u^{\mathrm{IV}}_{zT}(x,t)$$
$$= q_z(x,t) + m'_y(x,t), \tag{2.16.c}$$
$$\varrho I_{pT}(\ddot{\varphi}\dot{u}(x,t) - \varrho A(\ddot{u}_{zT}(x,t)y_T - \ddot{u}_{yT}(x,t)z_T) -$$
$$- \varrho I_\omega\varphi''(x,t) - I_tG\varphi''(s,t) + EI_\omega\varphi^{\mathrm{IV}}(x,t)$$
$$= m_{xT}(x,t) + m'_\omega(x,t), \tag{2.16.d}$$

where $0 \leq x \leq l$ and $0 \leq t \leq t_0$: boundary conditions

$$EAu'_{xT}(x,t) - \widehat{N}_{xT}(x,t) = 0, \tag{2.17.a}$$
$$I_{zz}\ddot{u}'_{yT}(x,t) - m_z(x,t) - EIu'''_{yT}(x,t) + \widehat{Q}_y(x,t) = 0, \tag{2.17.b}$$
$$EI_{zz}u''_{yT}(x,t) + \widehat{M}_z(x,t) = 0, \tag{2.17.c}$$
$$\varrho I_{yy}\ddot{u}'_{zT}(x,t) + m_y(x,t) - EI_{yy}u'''_{zT}(x,t) + \widehat{Q}_z(x,t) = 0, \tag{2.17.d}$$
$$EI_{yy}u''_{zT}(x,t) - \widehat{M}_y(x,t) = 0, \tag{2.17.e}$$
$$\varrho I_\omega\ddot{\varphi}'(x,t) + m_\omega(x,t) - EI_\omega\varphi'''(x,t) + I_tG\varphi'(x,t) - \widehat{M}_x(x,t) = 0 \tag{2.17.f}$$
$$EI_\omega\varphi''(x,t) + \widehat{B}(x,t) = 0\ , \tag{2.17.g}$$

where $x = 0$, or $x = l$, and $0 \leq t \leq t_0$, as well as initial conditions

$$\dot{u}_{xT}(x,0) - v^0_{xT}(x) = 0, \tag{2.18.a}$$
$$\dot{u}_{yT}(x,0) + z_T\dot{\varphi}(x,0) - [v^0_{yT}(x) + z_T\varkappa^0(x)] = 0, \tag{2.18.b}$$
$$u'_{yT}(x,0) - v'^0_{yT}(x) = 0, \tag{2.18.c}$$
$$\dot{u}_{zT}(x,0) - y_T\dot{\varphi}(x,0) - [v^0_{zT}(x) - y_T\varkappa^0(x)] = 0, \tag{2.18.d}$$
$$\dot{u}'_{zT}(x,0) - v'^0_{zT}(x) = 0, \tag{2.18.e}$$
$$\varrho I_{pT}[\dot{\varphi}(x,0) - \varkappa^0(x)] + \varrho A[(v^0_{zT}(x)y_T - v^0_{yT}(x)z_T)$$
$$- (\dot{u}_{zT}(x,0)y_T - \dot{u}_{yT}(x,0)z_T)] = 0, \tag{2.18.f}$$
$$\dot{\varphi}'(x,0) - \varkappa'^0(x) = 0, \tag{2.18.g}$$

where $0 \leq x \leq l$.

Initial conditions for $\dot{\varphi}(x,0)$, $\dot{u}_{yT}(x,0)$ and $\dot{u}_{zT}(x,0)$ may be simplified in a form by expressing terms $v^0_{zT}(x) - \dot{u}_{zT}(x,0)$ and $v^0_{yT}(x) - \dot{u}_{yT}(x,0)$, making use of (2.18.d) and (2.18.b), by means of $\dot{\varphi}(x,0) - \varkappa^0(x)$. The obtained (2.18.f) yields initial condition $\dot{\varphi}(x,0) - \varkappa^0(x) = 0$ yielding, in turn, initial conditions $\dot{u}_{yT}(x,0) - v^0_{yT}(x) = 0$ from (2.17.b), and $\dot{u}_{zT}(x,0) - v^0_{zT}(x) = 0$ from (2.18.d).

Relationships written for a single bar are easy to generalize for systems of interconnected bars, not to be detailed here.

Conclusions

Variation principles are efficient in research on mechanical problems. They throw a peculiar light on mechanical problems, likely to add momentum to further development. This is convincingly exemplified by their importance for the development of the finite element method (theoretical fundamentals and extension of sphere of applications).

Mathematical approach to variational principles makes functional analytic means available, underlying research and development of approximate (numerical) mathematical methods, indispensable in mechanics.

Another valuable feature of variational principles is their permitting integral, complex handling of mechanical problems, as pointed out by the method described above. Namely simultaneous treatment of motion equations, dynamical boundary conditions and initial conditions of velocity in a single relationship minimizes the possibility of mistakes.

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