NONPARAMETRIC IDENTIFICATION OF NONLINEAR ZADEH MODELS USING GAUSSIAN AUTOREGRESSIVE INPUT PROCESSES

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Summary

The paper presents a nonparametric identification method for the determination of the kernels of nonlinear analytic Zadeh models if the input signal is a Gaussian stationary autoregressive process.

Introduction

In recent times the nonlinear system-identification has gone through an important phase of development. The adaptation of modern mathematical tools [5] has led to the satisfactory principal solution of certain nonlinear differential equations using Volterra and analytical Zadeh functional series [7, 8]. The analytical Zadeh functional series, the Zadeh models make it possible ity to solve important types of nonlinear differential equation where the right side of the equations has a nonlinear character with unknown parameters typical cases of which are the nonlinear vibrating phenomena (e.g. in the modelling of vehicle system dynamics.) [9, 10].

In spite of the real importance of these nonlinear dynamic models, unfortunately, until now there are very few statistical results on their identification.

Therefore, the authors of this paper recently introduced the nonlinear analytic Zadeh model for the input/output representation of differential equation. [9, 10]. We dealt with the identification of these continuous Zadeh models where the input was a Gaussian white noise process.

Unfortunately for the active identification of nonlinear mechanical (dynamic) systems it is generally not possible to use even a bounded white noise as a test (input) signal. Similarly, the application of well-known pseudorandom signals is also problematic and insufficient. Therefore for the active identification of such nonlinear mechanical (dynamic) systems the low order Gaussian random processes seem to be an appropriate and effective test signal.

Thus, in this paper we consider the nonparametric identification of the continuous analytic Zadeh models in the time domain where the input is a first order autoregressive Gaussian random process (coloured noise) avoiding the difficulties of the more problematic and complicated frequency domain methods.

2. The Zadeh nonlinear model (and dynamic system) representation

It is well-known that the nonlinear system represented by Zadeh functional series is defined by equation

$$y(t)=\int_0^\infty u_0(s)ds+\sum_{i=1}^{n}\int_0^\infty\cdots\int_0^\infty u_i[x(t-s_1),\ldots,x(t-s_i),s_1\ldots,s_j]\,Jds_1\ldots ds_i+\xi(t). \tag{2.1}$$

Here the Zadeh kernels $u_i(x_1,\ldots,x_i,s_1\ldots s_i)$ will be considered as analytic, i.e

$$u_i(x_1,\ldots x_i,s_1,\ldots s_i)=\sum_{k\geq 1}a_k(s_1,s_2\ldots s_i)\,x^k$$

where $x^k=\prod_{l=1}^{i}x^{k_l}$ and moreover $\sum_{k\geq 1}(\prod_{l=1}^{i}k_l)\int_{R^n}a_k^2(s_1,\ldots,s_i)\,ds<\infty$ as well as the additive noise $\xi(t)$ is independent of input $x(t)$ and $E\xi(t)=0$.

The kernels $a_k(s_1,\ldots s_i)$ are called Raibman kernels of the Zadeh nonlinear model representation [8]. For the identification of the above kernels we assume that the following equations hold

$$a_k(s_1,s_2,\ldots,s_i)=a_l(\underbrace{s_1,\ldots,s_1}_{k_1};\underbrace{s_2,\ldots,s_2}_{k_2};\underbrace{s_i\ldots,s_i}_{k_i}) \tag{i}$$

where

$$k=(k_1,\ldots,k_i),\ l=\sum_{j=1}^{i}k_j,\ 1_l=(\underbrace{11\ldots 1}_{l})\text{ and }\prod_{r\neq p}(s_r-s_p)\neq 0,$$

$$a_k(s_1,\ldots,s_i)=a_{P_k}(Ps). \tag{ii}$$

Here P is an arbitrary permutation of the elements $1,2,\ldots,n$. The last condition can be satisfied by the summation $\sum_P a_{P_k}(Ps)$.

Because the weights a_k are coefficients of x^k thus the above conditions do not cause any loss of generality. In the next sections the input will be described by the Gaussian first order autoregressive (coloured) process and x^k will be replaced by Appel polynomials.

3. Multivariable Appel polynomial system

To approximate the analytic Zadeh nonlinear system in the case of a Gaussian autoregressive input let us introduce the n-variable Appel polynomial system [8].

1. $A_0 = 1$
2. $A(x_1, x_2, \ldots, x_n)$ is an n-variable symmetric polynomial of degree n in the variables $x_1, \ldots, x_n$;
3. $\frac{\partial}{\partial x_n} A_n(x_1, x_2, \ldots, x_n) = A_{n-1}(x_1, x_2, \ldots, x_{n-1})$;
4. $E\ A(x_1, x_2, \ldots, x) = 0$ where $x = (x_1, x_2, \ldots, x_n)$ is

a Gaussian vector-valued random variable with covariances $C_{x_i x_j}$ and $Ex = 0$.

If we denote the covariance of the variables x_i and x_j by $C_{x_i x_j}$ for the Appel polynomials $A_n(x)$ the following recursive formula holds (i.e. A_n can be calculated by the formula):

$$A_n(x_1, \ldots, x_n) = x_n A_{n-1}(x_1, \ldots, x_{n-1}) - \sum_{i=1}^{n-1} C_{x_i \cdot x_n} A_{n-2}(x_1, \ldots, x_{i-1} x_{i+1}, \ldots, x_{n-1}).$$

For the second order moments of the Appel polynomial system (in the case of the joint Gaussian distribution of the variables $x_1, \ldots x_n, z_1 \ldots z_m$ we get

$$EA_n(x_1, x_2, \ldots, x_n)\ A_m(z_1, \ldots, z_m) = \delta_{n,m} \sum_{n!}^{n}{}^{*} \prod_{l=1}^{n} C_{x_l z_{il}},$$

where the summation Σ^* is extended for all possible permutations $i_1, i_2, \ldots, i_n$ of numbers $1, 2, \ldots, n$. [7, 8].

Note that the system of the multivariable Appel polynomials ensures certain orthogonal properties to identify the analytic Zadeh model (Rajbman kernels) analogously to Wiener's G functionals for the identification of Volterra models (Wiener kernels) in the case of the stationary Gaussian input processes.

Thus the Appel polynomials have sufficiently general forms for containing the orthogonal structures as special cases to identify Wiener kernels of nonlinear systems represented by Volterra functional series. For example using the cross correlation function R_{yA_n} the Appel polynomials A_n are able to identify so called L functionals (both have an analogous structure [2]) given by Lee and Schetzen to identify Wiener kernels. If the input is a white Gaussian process it can provide the identification of Wiener kernels automatically through cross correlation R_{yA_n} [2]. (Naturally if the input is a white noise process then A_n is the product of $x(t - s_i)$, $i = 1, 2, \ldots l$.

If the input is a Gaussian white noise process, the Volterra model represented by G functionals can be described by the Appel polynomials too, i.e.

$$y(t) = \sum_{i=0}^{\infty} G_i[g_i, x(t)] = \sum_{i=0}^{\infty} \underbrace{\int \cdots \int}_{i} g_i(s_1, \ldots, s_i)\, A_i\, [(x(t - s_1), \ldots x(t - s_i)] =$$

$$= \sum_{j=0}^{\infty} \underbrace{\int \cdots \int}_{j} g_j^*(s_1, \ldots s_j) x(t - s_1) \ldots x(t - s_j) ds_1 \ldots ds_j.$$

Our propositions for the Appel polynomials hold also in the case when the variables are not different. Thus, we get the uni- and bivariable example, the bivariable Appel polynomial $A_{k,1}(x, y)$ is a $(k+1)$ — variable Appel polynomial, i.e.

$$A_{k,l}(x,y) = A_{k+l}(\underbrace{x, x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{l}) \tag{3.2}$$

As a generalization of this we introduce the following symbol

$$A_1k_2, \ldots k_s(x_1, \ldots, x_s) = A \sum_{i=1}^{s} k_j (\underbrace{x_1, \ldots, x_1}_{k_1}, \ldots \underbrace{x_s, \ldots, x_s}_{k_s}) \tag{3.3}$$

For the second order moments of Appel polynomials we get [8]

$$EA_{k_1,k_2,\ldots k_s}(x_1, \ldots x_s) A_{l_1}, \ldots {}_{l_p}(y_1, \ldots, y_p) =$$

$$= \delta_{k_1+\ldots+k_s}^{l_1+\ldots+l_p} \prod_{i=1}^{p} l_i! \sum_{\left\{\substack{\sum_{t=1}^{p} j_t^i = k_i \\ i=1,2,\ldots,s-1}\right\}} \binom{k_s}{(m_1, m_2, \ldots, m_p)} \prod_{i=1}^{p} C_{x_s y_i}^{m_i} \prod_{i=1}^{s-1}$$

$$\binom{k_i}{j_1^i, \ldots, j_p^i} \prod_{i=1}^{s-1} \prod_{l=1}^{p} C_{x_i y_l}^{j_l^i} \tag{3.4}$$

where $\quad 0 \leq m_i = l_i - \sum_{i=1}^{s-1} j_t^i \qquad$ and $\binom{k}{l_1, \ldots l_t} = \frac{k!}{\prod_{i=1}^{t} l_i!}$.

Using the multivariable Appel polynomials it can be seen that the equation of the analytical Zadeh nonlinear system (2.1) has the following equivalent form, i.e.

$$y(t) = \int_0^\infty u_0(s)ds + \sum_{i=1}^{\infty} \sum_{k \geq 1} \int_0^\infty \cdots \int_0^\infty a_k^*(s_1, \ldots, s_i) \prod_{l=1}^{i} x^{k_l}(t-s_l) ds_1, \ldots, ds_i + \xi(t) =$$

$$= \int_0^\infty u_0(s)ds + \sum_{l=1}^{\infty} \sum_{k \geq 1} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{i} a_k(s_1, \ldots, s_i) A_k[x(t-s_1), \ldots x(t-s_i)]\, ds_1 \ldots ds_i + \xi(t).$$

$$k = (k_1, \ldots, k_i) \tag{3.5}$$

4. Definition and application of the continuous AR(1) (coloured noise) process

Let us consider the Gaussian first order autoregressive stationary random process for the identification of the nonlinear dynamic models represented by the analytic Zadeh functional series. The first order autoregressive stochastic differential equation

$$dx(t) = -\alpha x(t)dt + cdw(t), \tag{4.1}$$

where $w(t)$ is a Wiener process given by the following spectral representation

$$w(t) = \int_{-\infty}^{\infty} \frac{e^{-i\lambda t} - 1}{i\lambda} S(d\lambda). \tag{4.2}$$

Here $S(d\lambda)$ is a Gaussian stochastic spectral measure. The autocorrelation function of $x(t)$ is

$$Ex(t)x(s) = R_{xx}(t-s) = \frac{c}{2\alpha} e^{-\alpha|t-s|}. \tag{4.3}$$

The autoregressive process $x(t)$ here will be denoted by $x_r(t)$ where $r = r(\alpha, c)$

Note:

It can be proved [8] that the process $x_r(t)$ converges (in weak topology) to the white noise process if α, $c \to \infty$ and $\alpha^2/c \to 1/2$.

This limit transition will be denoted by $r \to \infty$.

The computation of *Rajbman kernels* can be performed by cross correlation functions between the (centralized) output and the Appel polynomials ("on the white noise") i.e.

$$a_k(s_1, \ldots, s_j) = \frac{1}{j!} \lim_{r \to \infty} E(y(t) A_k^*[(z_r(t-s_1), \ldots, z_r(t-s_j))] =$$

$$= \frac{1}{j!} \lim_{r\to\infty} \Delta_r^2 R_{yA_k}(s) = \frac{1}{j!} E[y(t) A_k^*[x(t-s_1), \ldots, x(t-s_j)]] = \frac{1}{j!} R_{yA_k(x)}(s). \tag{4.4}$$

where

$$A_k^*[x(t-\tau_1), \ldots, x(t-\tau_i)] = \left(\frac{(2\alpha^{-1})^{\Sigma k_j - i}}{\prod_{j=1}^{i} (k_j - 1)!} \right)^{1/2} A_k[x(t-\tau_1), \ldots, x(t-\tau_i)] = \Delta_r A_k(\cdot)$$

and it can be proved that the process $\Delta_r A_n(Z_r)$ also converges to a non-Gaussian "white noise like" process [8].

For the multilinear case from (4.4) we get the well-known estimation of Wiener kernels determined by multivariable cross correlation functions as a special case [2, 8].

5. Nonparametric identification of continuous Zadeh models using first and second order autoregressive processes

For the nonparametric identification of the above Zadeh models let us consider the model equations (2.1) or (3.5) in the form

$$y(t)=\sum_{n=1}^{\infty}\sum_{k_1,\ldots,k_n\geq 1}\int\limits_0^{\infty}\cdots\int g_k(s)A_k(x_{t-s_1},\ldots,x_{t-s_n})\,ds+\xi(t), \tag{5.1}$$

where the Rajbman kernels have to satisfy further natural conditions (not losing their generality) in order to obtain a simpler approach of the nonparametric identification procedure.

Here let $x(t)$ be a Gaussian AR(1) input process with zero mean and unit variance (for the sake of simplicity) and let $\xi(t)$ be an additive noise independent from the input and $E\xi(t)=0$.

Furthermore A_k is an Appel polynomial with order n and degree $\sum_{i=1}^{n} k_i$.

Let us denote an arbitrary member of the Zadeh model in the following form:

$$y_k(t)=\int_0^{\infty}\cdots\int_0^{\infty} g_k(s)\,A_k(x_{t-s_1},\ldots,x_{t-s_n})\,ds. \tag{5.2}$$

Let π be a permutation of the numbers $1,2,\ldots,n$ and as well as

$$\pi(k)=(k_{\pi(1)},\ldots,k_{\pi(n)})$$

$$\pi(x_{t-s_1},\ldots,x_{t-s_n})=(x_{t-s_{\pi(1)}},\ldots,x_{t-s_{\pi(n)}}).$$

It can be seen that the Appel polynomials satisfy the following equation

$$A_k[\pi(x_{t-s_1},\ldots,x_{t-s_n})]=A_{\pi k}[(x_{t-s_1},\ldots,x_{t-s_n})].$$

Thus we may assume (without any restriction) that $k_1\leq k_2\leq k_n$. For the case $n=2$ if $k_1\neq k_2$

$$\iint_0^{\infty} g_{k_1,k_2}(s_1,s_2)A_{k_1,k_2}(x_{t-s_1},x_{t-s_2})\,ds=$$

$$=\iint_0^{\infty} g_{k_1,k_2}(s_2,s_1)\,A_{k_2,k_1}(x_{t-s_2},x_{t-s_1})\,ds=$$

$$=\iint_0^{\infty} g_{k_2,k_1}(s_2,s_1)A_{k_2,k_1}(x_{t-s_2},x_{t-s_1})\,ds.$$

i.e. we can achieve that there be only g_{k_1,k_2} kernels in the system. Because

$$\iint_0^{\infty} g_{k_1,k_2}(s_1,s_2)A_{k_1,k_2}(x_{t-s_1},x_{t-s_2})\,ds=$$

$$= \iint\limits_{s_1 \le s_2} + \iint\limits_{s_1 \ge s_2} g_{k_1,k_2}(s_1, s_2) A_{k_1,k_2}(x_{t-s_1}, x_{t-s_2}) \, ds =$$

$$= \iint\limits_{s_1 \ge s_2} g_{k_1,k_2}(s_1,s_2) A_{k_1,k_2}(x_{t-s_1}; \, x_{t-s_2}) \, ds +$$

$$+ \iint\limits_{s_1 \ge s_2} g_{k_2,k_1}(s_1, s_2) \, A_{k_2,k_1}(x_{t-s_1}, x_{t-s_2}) \, ds$$

without any restriction it may be assumed that

$$g_{k_1,k_2}(s_1,s_2) = 0 \qquad \text{if } s_1 \ge s_2 \text{ and}$$

$$g_{k_2,k_1}(s_1,s_2) = 0 \qquad \text{if } s_1 \ge s_2. \tag{5.3}$$

In the case of $k_1 = k_2$, the Appel polynomials are symmetric so the above condition holds automatically. This approach can be extended for any n and so we assume the following conditions for the kernels of the model; together with kernel g_k there is kernel $g_{\pi(k)}$ in the model:

$$g_k(\tau) = 0 \text{ if } \tau \in \{\tau | \tau_1 \le \tau_2 \le \ldots \le \tau_n\}. \tag{5.3}$$

Since the Appel polynomials of different degrees are orthogonal it is enough to consider only the covariances of Appel polynomials having the same finite degree (members with different orders are possible).

In this case the highest order Appel polynomial of Nth degree is $A_{11 \ldots 1}$ i.e. $k_1 = k_2 \ldots = k_n = 1$ ($n = N$) and this is the Nth order polynomial as well. So there is no other with an Nth degree and order N.

For the identification of kernel $g_{1 \ldots 1}$ it is necessary to determine the weighting function with the highest degree, and the appropriate member (which contains this kernel) will be excluded from the system, and so on.

If $g_{11 \ldots 1}(\tau) \neq 0$ then it is the member of highest degree, otherwise we have to indicate the rules of the choice of the member of highest degree.

The identification of $g_{11 \ldots 1}$ can be obtained from the cross correlation of the output and the Appel polynomials (defined on the prewhitened input process) i.e.

$$R_{yA^0_{1\ldots1}(e)}(\tau_1 \ldots \tau_N) = Ey(t) A_{1\ldots1}(e_{t-\tau_1}, \ldots, e_{t-\tau_N}) =$$

$$= \sum_{n=1}^{N-1} \sum_{\substack{n \\ \sum_{i=1}^{n} k_i = N}} \iint_0^\infty g_k(s) \, EA_k(x_{t-s_1}, \ldots, x_{t-s_n}) A_{1\ldots1}(e_{t-\tau_1}, \ldots, e_{t-\tau_N}) \, ds$$

$$+ \int_0^\infty \!\!\cdots\! \int g_{1\,1\ldots1}(s) EA_{1\ldots1}(x_{t-s_1}, \ldots, x_{t-s_N}) \cdot$$

$$\cdot A_{1\ldots1}(e_{t-\tau_1}, \ldots, e_{t-\tau_N}) \, ds. \tag{5.4}$$

Here we do not deal with the principal problem of the (i) prewhitening and (ii) the definition of the Appel polynomials on the white noise process, because (i) is a well-known problem and (ii) was discussed in detail in [7. 8.] by the authors of this paper.

We notice only that it is enough to prove the following limit relationships, since

$$E\{y(t)A_k[e(t-u)]\} = R_{yA_k}(u) = \lim_{r\to\infty} E\{y(t)A_k[x_r(t-u)]\} =$$

$$= \int_0^\infty g_k(s) \lim_{r\to\infty} E\{A_k[\int_0^\infty h_r(v)e(t-s-v)dv]A_k[x_{r_0}(t-u)]\}\, ds,$$

where h_r is the weighting function of the filter for ARMA's process $x_{r_0}(t)$, i.e.

$$x_{r_0}(t-s) = \int_0^\infty h_{r_0}(v)e(t-s-v)dv$$

we get that

$$\lim_{r\to\infty} E\{A_k[x_r(t-s)]A_k[x_{r_0}(t-u)]\} = \lim_{r\to\infty} R^k_{x_r x_{r_0}}(u-s)$$

$$= \lim_{r\to\infty} \{\delta_{u\geq s} \int_0^\infty h_{r_0}(u-s+v)h_r(v)dv = h_{r_0}(u-s)\delta_{u\geq s}$$

from where we obtain the "expected" result, i.e.

$$R_{yA_k^0}(u) = \int_0^u g_k(s)h^k_{r_0}(u-s)ds\ .$$

In the discrete case, naturally, we have no theoretical problems of the above type.

Thus it is sufficient to use the covariance

$$\operatorname{cov}(x_s, e_t) = \begin{cases} e^{-\alpha(s-t)} & s \geq t \\ 0 & s < t \end{cases}$$

and the covariance theorem for Appel polynomials with a slight modification according to the formula (3.4).

$$EA_{k_1,\dots,k_n}(x_1,\dots,x_n)A_{l_1,\dots,l_m}(z_1,\dots,z_m) =$$

$$= \delta^{\Sigma l_i}_{\Sigma k_i} \prod_{i=1}^m l! \sum_{\substack{\sum_p j_p^q = k_q \\ \sum_q j_p^q = l_p \\ j_p^q \geq 0}} \prod_p \binom{k_p}{j_1^p, \dots, j_m^p} \prod_{p=1}^n \prod_{q=1}^m C^{j_q^p}_{(x_p, z_q)}$$

where $C(x_p, z_q)$ is the covariance between the random variables x_p and z_q.

Taking into consideration that $g_k(s) \neq 0$ if $s_1 \leq s_2 \leq \dots \leq s_n$ we get that

$$R_{yA_{11\dots1(e)}}(\tau_1,\tau_2\dots\tau_N)=$$

$$=\sum_{n-1}^{N-1}\sum_{\sum_i^n k_i=N}\pi_{l_i}!\sum_{\begin{Bmatrix}\sum_q j_p^q=k_p\\ \sum_p j_p^q=l_q\\ j_p^q\geq 0\end{Bmatrix}}\prod_{p=1}^{n}\binom{k_p}{j_1^p,\dots,j_m^p}\int_0^{w_1}\dots\int^{w_n}g_k(s)e^{-\alpha\sum_{pq}j_p^q(\tau_q-s_p)}\,ds+$$

$$+\sum_{N!}\int_0^{\tau_{q_1}}\dots\int_0^{\tau_{qN}}g_{1\dots1}(s)\,e^{-\alpha\sum_p(\tau_{qi_p}-s_p)}\,ds.$$

The summation in the last additive member of the above formula is valid for the entire π permutation of numbers $1\dots N$: $\pi(1,\dots,N)=(q_1,\dots q_N)$ and since the property (ii) of the kernels g the upper bound of the i-th integral will be min τ_{p_j}. Then, since $\tau_1\geq\dots\geq\tau_n$ there is no repetition in the upper bounds τ_i only for the case when $\pi(1,\dots,N)=(1,\dots,N)$.

In the first additive member of the formula (5.6) $w_1,\dots,w_n$ denotes the appropriate bounds of the integrals. Let us now analyse the influence of the operator

$$L=\prod_1^N\left(\alpha+\frac{\partial}{\partial\tau_i}\right)\tag{5.7}$$

on cross correlation $R_{yA_1^e}\dots{}_1(\tau)$. First, considering the operator $(\alpha+\delta/\delta\tau_N)$, could be only in the last place, among the upper bounds of the integrals because the property (ii) of kernels g (otherwise the larger one would follow it) the upper bounds of integrals follow a monoton series.

If $w_n=\tau_N$ and $\tau_{p_N}\neq\tau_N$ then

$$\left(\alpha+\frac{\partial}{\partial\tau_N}\right)\int_0^{w_1}\dots\int_0^{w_n}g_k(s)e^{-\alpha\sum_q\sum_p j_p^q(\tau_q-s_p)}\,ds=$$

$$=\alpha\int_0^{w_1}\dots\int_0^{w_n}g_k(s)e^{-\alpha\sum_q\sum_p j_p^q(\tau_q-s_p)}\,ds-$$

$$-\alpha\underbrace{\sum_{p=1}^{n}j_p^N}_{=1}\int_0^{w_1}\dots\int_0^{w_n}g_k(s)e^{-\alpha\sum_q\sum_p j_p^q(\tau_q-s_p)}\,ds=0.\tag{5.8}$$

If τ_N is the upper bound of the integral in any additive member then

$$\left(\alpha+\frac{\partial}{\partial\tau_N}\right)\int_0^{w_1}\dots\int_0^{\tau_N}g_k(s)\,e^{-\alpha\sum_{p=1}^{n}\sum_{q=1}^{N}j_p^q(\tau_q-s_p)}\,ds=$$

$$= \int_0^{w_1} \cdots \int^{w_{n-1}} g_k(s_1, \ldots, s_{n-1}, \tau_N) e^{-\alpha \sum_{t=1}^{n} \sum_{q=1}^{N-1} j_p^q(\tau_q - s_p)} \mathrm{d}s. \tag{5.9}$$

This is valid if $j_n^N = 1$ and $j_k^N = \emptyset$, $k > 1, \ldots n-1$ and therefore $k_n = 1$. But if $n < N$ there is at least one $k_i > 1$ because of $\sum_1^n k_i = N$.

Thus it is clear that during the further influence of the operator, only one member remains in the sum when $k_i = 1$, $i = 1, \ldots, N$ or $p_i = i$.

Now the member containing the kernel $g_{1 \ldots 1}$ i.e.

$$y_{1\ldots1}(t) \int_0^\infty \cdots \int_0 = g_{1\ldots1}(s)\, A_{1\ldots1}(x_{t-s_1}, \ldots, x_{t-s_p}) \mathrm{d}s \tag{5.10}$$

is excluded from $y(t)$.

Thus, the model obtained contains no higher order members than $(N-1)$th order ones. The further determination of the kernels can be carried out by induction. Therefore let us assume that the model does not contain members of order $n+1$ and degree N. Let us determine and neglect the kernels with order n and degree N, and so on. The computation of the kernels of order n can be realised also by induction. The choice can be "carried out" by the following algorithm: Let K be the set of kernels of order n and degree N "belonging" to the system, moreover:

$$\begin{aligned}
K_1 &= \{k \in K | k_1 = \max_{l \in K} l_1\} \\
K_{1,n} &= \{k \in K_1 | k_n = \max_{l \in K_1} l_n\} \\
K_2 &= \{k \in K_{1,n} | k_2 = \max_{l \in K_{1,n}} l_2\} \\
K_{2,n-1} &= \{k \in K_2 | k_{n-1} = \min_{l \in K_2} l_{n-1}\} \\
K_{\left[\frac{n}{2}\right]} &= \{k \in K_{\left[\frac{n}{2}\right]-1, n-\left[\frac{n}{2}\right]} | k_{\left[\frac{n}{2}\right]} = \max_{l \in K_{[n/2]-1, n-[n/2]}} l_{\left[\frac{n}{2}\right]}\} \\
K_{\left[\frac{n}{2}\right], n-\left[\frac{n}{2}\right]+1} &= \{k \in K_{\left[\frac{n}{2}\right]} | k_{n-\left[\frac{n}{2}\right]+1} = \min_{l \in K_{[n/2]}} l_{n-\left[\frac{n}{2}\right]+1}\}
\end{aligned} \tag{5.11}$$

If n is even then the last set $K_{\left[\frac{n}{2}\right], n-\left[\frac{n}{2}\right]+1}$ and if n is odd then the last one

$$K_{\left[\frac{n}{2}\right]+1} = \{k \in K_{\left[\frac{n}{2}\right], n-\left[\frac{n}{2}\right]+1} | k_{\left[\frac{n}{2}\right]+1} = \min_{l \in K_{[n/2], n-[n/2]+1}} l_{\left[\frac{n}{2}\right]+1}\}.$$

This set series is decreasing monotonically and has only one common element . The cross correlations R_{yA_k} if $\tau_1 \leq \ldots \leq \tau_N$ is

$$R_{yA_k(e)}(\tau_1, \ldots, \tau_n) = \sum_{m=1}^{n} \sum_{\substack{m \\ \sum_1 l_i = N}} \int_0^\infty \cdots \int g_l(s) E A_l(x_{t-s_1}, \ldots, x_{t-s_m}) A_k(e_{t-\tau_1}, \ldots, e_{t-\tau_m}) \mathrm{d}s$$

and so

$$R_{yA_k(e)}(\tau_1 \dots \tau_n) = \sum_{m=1}^{n} \sum_{l|\sum^m l_i = N} \sum_{\substack{\sum_q j_p^q = l_p \\ \sum_p j_p^q = k_p}} \prod_{p=1}^{n} k_p! \prod_{q=1}^{m} \binom{l_q}{j_q^p \dots j_q^m} \cdot$$

$$\cdot \int_0^{w_1} \dots \int^{w_m} g_l(s) e^{-\alpha \sum_{p,q} j_p^q(\tau_q - s_p)} \, ds \qquad (5.12)$$

where the upper bounds of the inregral are from the appropriate τ_i, $i = 1, 2, \dots$ Let us analyse the case when

$$W_m = \tau_n, \; W_{m-1} = \tau_{n-1}, W_1 = \tau_{n-m+1}.$$

Taking into consideration the property (ii) of the kernels g and the cross correlation between the original input and the prewhitened input the following equalities hold

$$W_m = \tau_n \text{ if } j_m^n \neq 0 \text{ and } j_m^{n-1} = j_m^{n-2} = \dots = j_m^1 = 0$$

which means, that

$$j_m^n = l_m; \; l_m \leq \sum_q j_m^q = k_n$$

$$W_{m-1} = \tau_{n-1} \text{ if } j_{m-1}^{n-1} \leq 0, \; j_{m-1}^{n-2} = \dots = j_{m-1}^1 = 0$$

i.e.

$$j_{m-1}^n + j_{m-1}^{n-1} = l_{m-1}$$

$$\vdots$$

$$W_1 = \tau_{n-m+1} \text{ if } j_1^{n-m+1} \neq 0, \; j_1^{n-m} = 0 \dots = j_1^1 = 0$$

but then $j_q^1 = 0$, $q = 1, 2, \dots m$
and so $\sum_p j_p^1 = k_1 = 0$ that is a contradiction when $m < n$. If $m = n$, $W_m = \tau_n$ if $j_n^n \neq 0$ and $j_n^{n-1} = \dots = j_n^1 = 0$ and so since $j_n^n = l_n$, $l_n \leq k_n$ but from minimum property of k_n it holds that $k_n = l_n$

$$W_{n-1} = \tau_{n-1} \text{ if } j_{n-1}^{n-1} \neq 0, \; j_{n-1}^{n-2} = \dots = j_{n-1}^1 = 0$$

$$j_{n-1}^n + j_{n-1}^{n-1} = l_{n-1}$$

$$\vdots$$

$$W_1 = \tau_1 \text{ if } j_1^1 = 0.$$

Thus $j_1^1 = k_1 \leq \sum_q j_1^q = l_1$

and because of the maximality of k_1 we get that $k_1 = l_1$ and $k_n = l_n$ is possible and only if $j_1^q = \emptyset$, $q = 2, \dots$ moreover $j_p^n = 0$, $p = 1, 2, \dots, n-1$.

Thus $j_{n-1}^{n-1} = l_{n-1}$ and $j_2^2 = k_2$

from where $l_{n-1} \leq k_{n-1}$ and $l_2 \geq k_2$ but for the minimum property of k_{n-1} as well as for the maximum property of k_2, $k_{n-1} = l_{n-1}$ and $l_2 = k_2$ and so on.

As a consequence we obtain that $W_1 = \tau_1, \dots W_n = \tau_n$ is possible if and only if $k = l$.

But in this case

$$\prod_{i=1}^{n} \left(\alpha k_i + \frac{\partial}{\partial \tau_i}\right) R_{yA_k(e)}(\tau_1, \dots, \tau_n) = \prod_{p=1}^{u} k_p!\, g_k(\tau)$$

i.e. the formula for the determination of the Rajbman kernels of the analytic Zadeh model can be obtained in the following way

$$\prod_{i=1}^{n} \frac{1}{k_i!} \left(\alpha k_i + \frac{\partial}{\partial \tau_i}\right) R_{yA_k(e)}(\tau) = g_k(\tau).$$

Thus, using the above expression, if the input is an AR(1) test process, the Rajbman weighting functions can be computed by a relatively simple method. Naturally we may obtain entirely similar results for discrete (or discretised) input/output processes as well.

Finally let us consider the case when the input is a Gaussian second order autoregressive stationary process with zero mean, i.e.

$$\ddot{x}(t) = \alpha \dot{x}(t) + \beta x(t) + e(t)$$

or in the frequency domain the transfer function of filleris

$$\mathrm{H}(s) = \frac{1}{-\beta - \alpha s + s^2}$$

where s is the variable of Laplace's transformation and λ_1, λ_2 are the roots of the equation $-\beta - \alpha s + s^2 = 0$ and so the transfer function of the filter is

$$H(s) = \frac{1}{(\lambda_1 + s)(\lambda_2 + s)} = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{1}{\lambda_1 + s} - \frac{1}{\lambda_2 + s}\right)$$

from where the weighting function of the filter is

$$h(\tau) = \frac{1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 \tau} - \bar{e}^{\lambda_2 \tau}).$$

It can be proved easily that the transfer function of $h^n(s)$

$$H_n(s) = H[h^n(\tau)] = \frac{1}{(\lambda_2 - \lambda_1)^h} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(n-k)\lambda_1 + k\lambda_2 + s} =$$

$$= \frac{n!}{\prod_{k}^{n} (n-k)\lambda_1 + k\lambda_2 + s}.$$

Let

$$\prod_{j=1}^{n} \prod_{i=1}^{k_j} \left\{(n - k_j)\lambda_1 + k_j\lambda_2 + \frac{\partial}{\partial \tau_i}\right)\right\} = P_2.$$

Using the operator P_2 we can prove the following formula for the determination of the Rajbman kernels if the input is an AR(2) process (according to the above algorithm—in the case of the AR(1) input);

$$P_2\{R_{yA_k}(\tau)\} = n!\; g_k(\lambda) \prod_{j=1}^{n} k_j!\,.$$

Remark: Naturally the above method can be used also for the identification of Volterra kernels according to the results obtained by Schetzen in the frequency domain [2]. Thus

$$R_{yee\ldots e}(u_1, \ldots, u_i) = \underbrace{\int_0^\infty \ldots \int_0^\infty}_{i} a(s_1, \ldots, s_i) h(u_1 - s_1), \ldots, h(u_i - s_i) ds_1 \ldots ds_i$$

where the Volterra (Wiener) kernels a $(\tau_1, \ldots, \tau_i)$ can be computed by formula: when the input is an AR(1) process

$$\frac{1}{n!} \prod_{j=1}^{i} \left(\alpha + \frac{\partial}{\partial \tau_j}\right) R_{ye\cdots e}(\tau_1 \ldots, \tau_i) = a(\tau_1 \ldots, \tau_i).$$

If the input is an AR(2) one

$$a(u_1, \ldots, u_2) = \frac{1}{n!} \prod_{j=1}^{i} \left(\lambda_1 + \frac{\partial}{\partial u_j}\right) \left(\lambda_2 + \frac{\partial}{\partial u_j}\right) R_{ye\ldots e}(u_1, \ldots, u_i).$$

6. Important special cases

Let us now consider the application of the above results . . . for a second . . . order Zadeh model (with second degrees).

The determination of the second order kernel $a_{22}(\ldots)$ can be obtained by the formula

$$a_{22}(\tau, \sigma) = \frac{1}{2} \left(2\alpha + \frac{\partial}{\partial \tau}\right) \left(2\alpha + \frac{\partial}{\partial \sigma}\right) R_{yA_{22}^0}(\lambda, \sigma)$$

where

$$R_{yA_{22}^0}(\tau, \sigma) = E\{y(t) A_2[e(t-\tau)] A_2[e(t-\sigma)]\}.$$

It can be noticed that if there were a second order kernel $a_{23}(\ldots)$ as well then,—according to the algorithm introduced we first have to chose the kernel

$$3a_{13}(\tau^7, \tau_2) = \left(\alpha + \frac{\partial}{\partial \tau_1}\right) \left(3\alpha + \frac{\partial}{\partial \tau_2}\right) R_{yA_{13}^0}(\tau_1, \tau_2) \tag{6.2}$$

and

$$R_{yA_{13}^0}(\tau_1, \tau_2) = E(y(t) A_1[e(t-\tau_1)] A_2[e(t-\tau)]\,.$$

In this case the computation of the kernel $a_{22}(\ldots)$ can be carried out by the expression

$$2a_{22}(\tau, \sigma) + 3a_{13}(\tau, \sigma) e^{-\alpha(\tau, \sigma)} = \left(2\alpha + \frac{\partial}{\partial \tau}\right) \left(2\alpha + \frac{\partial}{\partial \sigma}\right) R_{yA_{22}^0}(\tau, \sigma)\,.$$

Furthermore

$$2a_{11}(\tau,\sigma) = \left(\alpha + \frac{\partial}{\partial\tau}\right)\left(\alpha + \frac{\partial}{\partial\sigma}\right) R_{yA_{11}}(\tau,\sigma)$$

$$a_{21}(\tau,\sigma) = \left(\alpha + \frac{\partial}{\partial\tau}\right)\left(2\alpha + \frac{\partial}{\partial\sigma}\right) = R_{yA_{12}}(\tau,\sigma). \tag{6.3}$$

For the one dimensional kernels of second order Zadeh models ($n = 2$) we get

$$a_1(\tau) = \left(\alpha + \frac{\partial}{\partial\tau}\right) R_{yA_1^\circ}(\tau)$$

and

$$a_2(\tau) = \left(2\alpha + \frac{\partial}{\partial\tau}\right) [R_{yA_2^\circ}(\tau) - \sqrt{2}\underset{\tau>\sigma}{\int_0^\tau\int_0^\tau} a_{11}(\tau,\sigma)e^{-2\alpha(\tau+\sigma)}d\tau d\sigma] \tag{6.4}$$

when assuming that $a_3(.) = a_4(.) = \ldots = 0$

Analogously, if the input is a Gaussian AR(2) stationary process then for example for the kernel $a_{22}(.,.)$ we obtain the following similar result

$$\left(2\lambda_1 + \frac{\partial}{\partial v}\right)\left(\lambda_1 + \lambda_2 + \frac{\partial}{\partial v}\right)\left(2\lambda_2 + \frac{\partial}{\partial v}\right)\left(2\lambda_1 + \frac{\partial}{\partial u}\right)\cdot$$

$$\cdot\left(\lambda_1 + \lambda_2 + \frac{\partial}{\partial u}\right)\left(2\lambda_2 + \frac{\partial}{\partial u}\right) R_{yA_{22}^\circ}(u,v)) = D_{22}^{(2)}R_{yA_{22}^\circ}(u,v) = 2a_{22}(u,v) \tag{6.5}$$

because

$$R_{yA_{22}(e)}(u,v) = 2\int_0^u\int_0^v a_{22}(s,z)\frac{1}{(\lambda_2-\lambda_1)^4}\cdot[(e^{-2\lambda_1(u-s)} - 2e^{-(\lambda_1+\lambda_2)(u-s)} + e^{-2\lambda_2(u-s)}$$

$$\cdot(e^{-\lambda_1(v-z)} - e^{-\lambda_2(u-z)})]^2 ds dz + \frac{4}{(\lambda_2-\lambda_1)^4}\int_0^v\int_0^v a_{22}(u,s)[e^{-\lambda_1(u-s)} - e^{-\lambda_2(u-s)})]\cdot$$

$$\cdot[e^{-\lambda_1(u-z)} - e^{-\lambda_2(u-z)})]\,[(e^{-\lambda_1(v-z)} - e^{-\lambda_2(v-z)})]\cdot[e^{-\lambda_1(v-z)} - e^{\lambda_2(v-z)}]dsdz\,.$$

For practical applications let us compute the Rajbman kernels of the second order Zadeh model from discretised samples of the ergodic stationary Gaussian I/0 processes.

In this case the equation of the discrete second order Zadeh model is

$$y_t = \sum_{i=0}^{\infty} a_1(i)A_1(x_{t-i}) + \sum_{\substack{i,j=0\\ i>j}}^{\infty} a_{11}(h,j)A_{11}(x_{t-i}, x_{t-j}) + \sum_{i=0}^{\infty} a_2(i)\frac{1}{\sqrt{2}}A_2(x_{t-i}) +$$

$$+\sum_{i>j}^{\infty} a_{1,2}(i,j)\,A_{1,2}(x_{t-i}, x_{t-j})\frac{1}{\sqrt{2}} + \sum_{i>j} a_{21}(i,j)\,A_{21}(x_{t-i}, x_{t-j})\frac{1}{\sqrt{2}} +$$

$$+\sum_{\substack{i,j \\ i>j}} a_{22}(i,j) A_{22}(i,j) \frac{1}{2}. \tag{6.6}$$

If the weighting coefficients of the filter of the input process are c_p ($p = 1, 2 \ldots.$) then the polynomial cross correlation function between the input filter (source) white noise and the output process of the second order Zadeh model has the form:

$$R_{yA_1^0}(p) = \sum_{i=0}^{p} a_1(i) c_{p-i} \tag{6.7}$$

$$R_{yA_{11}^0}(p,q) = \sum_{i>j}^{p,q} a_{11}(i,j)\, c_{p-i} c_{q-j} + \sum_{i>j}^{q} a_{11}(i,j) c_{p-j}\, c_{g-i},\ p > q$$

and neglecting the kernel $a_{11}(i, j)$, $i < j$

$$R_{yA_2^0}(p) = \sum_{i=0}^{p} a_2(i) c_{p-i}^2 \tag{6.8}$$

and

$$R_{yA_2^0}(p) = \sum_{i=0}^{p} a_2(i) c_{p-i}^2 .$$

If the input is an AR(1) process i.e. $x_t + \varrho x_{t-1} = e_t$ (where e_t $N(0,1)$) and $|\varrho| < 1$, furthermore $P_1(z^{-1}) = 1 + \varrho z^{-1}$ we get the following relationships ($P_1/z^{-1}/c_j = \delta_j = 0,\ c_j = (-\varrho)^j$) if z is

$$P_2(Z^{-1}) = 1 - \varrho^2 Z^{-1} \tag{6.9}$$

$$P_2(Z^{-1}) \hat{R}_{yA_2^0}(p) = a_2(p)$$

$$R_{yA_2^0}(p) = R_{yA_2^0}(p) - \frac{2}{\sqrt{2}} \sum_{i>j}^{p} \sum^{p} a_{11}(i,j) \varrho^{2p-(i+j)}$$

$$P_1(z_1^{-1}) P_2(z_2^{-1}) R_{yA_{11}^0}(p,q) = a_{11}(p,q)$$

$$P_1(z_1^{-1}) P_2(z_2^{-1}) R_{yA_{12}^0}(p,q) = a_{12}(p,q) \qquad p > q$$

$$P_2(z_1^{-1}) P_1(z_2^{-1}) \widehat{R_{yA_{21}^0}}(p,q) = a_{21}(p,q) \qquad p > q$$

$$P_2(z_1^{-1}) P_2(z_2^{-1}) R_{yA_{22}^0}(p,q) = a_{22}(p,q) \qquad p \neq q .$$

In spite of the continuous case the powers of the discrete AR(2) (input) processes are not autoregressive processes and so the identification of the Rajbman kernels of Zadeh model (6.6) in the time domain can be carried out by more complicated methods, but that will be the subject of a forthcoming paper.

Conclusions

The paper introduced a method for the determination of analyticcal Zadeh models. The result obtained can be used for the nonparametric identification of the nonlinear systems represented by the analytic Zadeh functional series and the structural estimation of nonlinear dynamic systems described by a certain class of nonlinear statistic differential equations.

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