THE INFLUENCE OF SOME NONLINEARITIES FOR THE DESIGN OF VEHICLE STRUCTURES

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Summary

Design of vehicle structures often involves the following problems:

— analysis of structural members with nonlinear stiffness characteristics (e.g. members behaving differently in tension and in compression when the effect of tipping superstructures is reckoned with);
— analysis of stability problems (e.g. consideration of lattice bars in autobus sidewalls);
— deliberate modification of stiffness characteristics upon inadequate outcomes (insufficient or excessive strength).

A matrix equation for successive modifications offers an economical solution of these rather simple nonlinear problems as a series of linear problems.

Introduction

Solution of a hyperstatic problem with several redundancies yields for stresses in the hyperstatic structure [1]:

$$\mathbf{L} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^T\mathbf{R}\mathbf{A} = (\mathbf{E} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^T\mathbf{R})\mathbf{A}$$

where:

$\mathbf{D}^{-1}\,\mathbf{B}^T\,\mathbf{R}\,\mathbf{A} = -\mathbf{X}$ — matrix of connecting forces, thus, concisely:

$$\mathbf{L} = \mathbf{A} + \mathbf{B}\mathbf{X}$$

where:

$\mathbf{A}$	internal loads of the basic system due to an outer load;
$\mathbf{B}$	internal loads of the basic system due to unit loads;
$\mathbf{R}$	flexibility matrix comprising stiffness characteristics;
$\mathbf{D} = \mathbf{B}^T\,\mathbf{R}\mathbf{B}$	connection displacements due to unit loads;
$\mathbf{D}_0 = \mathbf{B}^T\,\mathbf{R}\mathbf{A}$	connection displacements due to an outer load.

Be member J of the structure of limited load capacity imposed an ultimate stress ϵ_J. Calculations above have yielded for structural member J:

$$\mathbf{L}_J = \mathbf{A}_J + \mathbf{B}_J\mathbf{X}_J \leq \epsilon_J$$

where $\mathbf{A}_J$, $\mathbf{B}_J$, $\mathbf{X}_J$ are minormatrices of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$, resp., relating to member J.

To ease computations, generally a transforming matrix $\mathbf{T}$ can be found, yielding, after transforming the basic system as $\mathbf{x} = \mathbf{T}^T\mathbf{y}$ [2], [3]:

$$\mathbf{L}_J = \mathbf{E}\,\mathbf{y}_J = \mathbf{y}_J \leq \boldsymbol{\epsilon}_J$$

that is, the new basic system is cut exactly at members of limited load capacity, where: $\mathbf{A}_J = \mathbf{0}$, $\mathbf{B}_J = \mathbf{E}$.

In the relationship for the new basic system:

$$\mathbf{y}_J - \boldsymbol{\epsilon}_J \leq \mathbf{0}$$

specifying $\boldsymbol{\epsilon}_J = \mathbf{0}$, $\mathbf{y}_J \leq \mathbf{0}$ yields the effect of a one-way connection (absorbing compressions alone), while specifying $\boldsymbol{\epsilon}_J < \mathbf{0}$ leads to the solution of the stability problem.

In solving the stability problem, data of $\boldsymbol{\epsilon}_J$ comprise critical forces in the members.

In final account, a plastic hinge in the structure leads to a similar problem, irrespective of the gradual formation of the plastic hinge, and considering only the ultimate condition as critical in approximate calculations.

Analysis of structural members with nonlinear stiffness characteristics

Energy accumulated in the examined structure:

$$W(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{D}\mathbf{x} + \mathbf{d}_0^T\mathbf{x} + \gamma$$

Let us find the minimum of $W(\mathbf{x})$ under condition:

$$x_i \leq 0 \qquad i \in J.$$

Its physical purport makes matrix $\mathbf{D}$ a symmetric, positive definite matrix.

The minimization problem is a quadratic programming problem strictly convex from below, with a minimum point $\mathbf{x}^*$ given by the Kuhn—Tucker theorem necessary and sufficient conditions such as:

Be I the set of subscripts with no limiting condition for variable x_i, and be $\mathbf{x}_I$ the vector of these variables. Be J subscripts referring to variables meeting condition $x_j \leq 0$, and be $\mathbf{x}_J$ vector of these variables.

Writing matrix $\mathbf{D}$, as well as vectors $\mathbf{d}_0$ and $\mathbf{x}$ according to blocks corresponding to subscript sets I and J:

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{II} & \mathbf{D}_{IJ} \\ \mathbf{D}_{JI} & \mathbf{D}_{JJ} \end{bmatrix}; \quad \mathbf{d}_0 = \begin{bmatrix} \mathbf{d}_{0I} \\ \mathbf{d}_{0J} \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_I \\ \mathbf{x}_J \end{bmatrix}$$

where $\mathbf{D}_{IJ} = \mathbf{D}_{JI}^T$.

With these notations, the energy cumulated in the structure is:

$$W(\mathbf{x}_I, \mathbf{x}_J) = \frac{1}{2}\mathbf{x}_I^T\mathbf{D}_{II}\mathbf{x}_I + \mathbf{x}_I^T\mathbf{D}_{IJ}\mathbf{x}_J + \frac{1}{2}\mathbf{x}_J^T\mathbf{D}_{JJ}\mathbf{x}_J + \mathbf{d}_{0I}^T\mathbf{x}_I + \mathbf{d}_{0J}^T\mathbf{x}_J + \gamma$$

Lagrangian function for the minimization problem:

$$L(\mathbf{x}_I, \mathbf{x}_J, \boldsymbol{\lambda}) = W(\mathbf{x}_I, \mathbf{x}_J) + \boldsymbol{\lambda}_J^T\mathbf{x}_J$$

According to the Kuhn—Tucker theorem, if $\mathbf{x}^* = (\mathbf{x}_I^*, \mathbf{x}_J^*)$ is the solution of the minimization problem, then a Lagrangian multiplier $\boldsymbol{\lambda}_J^* = \mathbf{0}$ exists, such that:

$$L'_{x_I}(\mathbf{x}_I^*, \mathbf{x}_J^*, \boldsymbol{\lambda}_J^*) = 0$$
$$L'_{x_J}(\mathbf{x}_I^*, \mathbf{x}_J^*, \boldsymbol{\lambda}_J^*) = 0$$
$$\boldsymbol{\lambda}^{*T}L'_{\lambda}(\mathbf{x}_I^*, \mathbf{x}_J^*, \boldsymbol{\lambda}_J^*) = 0$$
$$\mathbf{x}_J \leq \mathbf{0}$$

leading, in the actual case, to the set of conditions:

$$\mathbf{D}_{II}\mathbf{x}_I^* + \mathbf{D}_{IJ}\mathbf{x}_J^* + \mathbf{d}_{0I} = \mathbf{0}$$
$$\mathbf{D}_{JI}\mathbf{x}_I^* + \mathbf{D}_{JJ}\mathbf{x}_J^* + \mathbf{d}_{0J} = -\boldsymbol{\lambda}_J^* \leq \mathbf{0}$$
$$\mathbf{x}_J^* \leq \mathbf{0}, \ \boldsymbol{\lambda}_J^{*T}\mathbf{x}_J^* = 0$$

In the actual case, the Lagrangian multiplier $\boldsymbol{\lambda}_J$ has a concrete physical meaning: deformations of structural members under the limiting condition.

Remind that sign conventions for $\mathbf{x}_J^*$ and $\boldsymbol{\lambda}_J^*$ impose condition $\boldsymbol{\lambda}_J^{*T}\mathbf{x}_J^* = 0$ to be met for each coordinate, i.e.:

$$\lambda_j x_j = 0 \qquad \forall_j \in J.$$

This latter is the so-called condition of *complementarity*, namely if there is a sign limitation for variable x_j, then this latter, or coordinate j of the objective function gradient, is zero.

This phenomenon is of use in generating the solution algorithm.

Let us introduce sets of subscripts:

$$J^-(x) = \{i \in J \quad : \ x_i < 0\}$$
$$J^\circ(x) = \{i \in J \quad : \ x_i = 0\}$$
$$J^+(x) = \{i \in J \quad : \ x_i > 0\}$$
$$J_+^\circ(x) = \{i \in J^\circ(x) \quad : \ \mathbf{d}_i^T\mathbf{x} + d_{0i} > 0\}$$
$$J_-^\circ(x) = \{i \in J^\circ(x) \quad : \ \mathbf{d}_i^T\mathbf{x} + d_{0i} \leq 0\}$$

where $\mathbf{d}_i$ is ith vector of matrix $\mathbf{D}$.

Steps for solving the minimization problem are:

1. $$k = 0,\ \tau_0 = R^n$$

2. $$\bar{\mathbf{x}}^k = \underset{\tau_k}{\text{argmin}}\ W(x)$$

3. $$x_i^k = \begin{cases} \bar{x}_i^k, & \text{for } i \in J^-(\bar{\mathbf{x}}^k) \cup J^\circ(\bar{\mathbf{x}}^k) \cup I \\ 0, & \text{for } i \in J^+(\bar{\mathbf{x}}^k) \end{cases}$$

4. $$\text{If } I \cup J^-(\mathbf{x}^k) = 0 \text{ or}$$

$$\mathbf{d}_i^T \mathbf{x}^k + d_{0i} = 0 \quad \forall i \in I \cup J^-(\mathbf{x}^k)$$

then the fifth, otherwise the seventh step is:

5. $$\text{If}\quad \mathbf{d}_i^T \mathbf{x}^k + d_{0i} \leq 0 \quad \forall i \in J^\circ(\mathbf{x}^k) \quad \text{then}$$

$$\mathbf{x}^* = \mathbf{x}^k \qquad \text{results, otherwise}$$

$$\mathbf{Y}kj = \mathbf{x}^k \qquad \text{and}$$

6. $$\tau_{k+1} = \{\mathbf{x} \in R^n : x_i = 0,\ \text{ for } \ i \in J_-^\circ(\mathbf{x}^k)\}$$

$k = k + 1$ and computation has to be resumed with step 2.

7. $$\tau_{k+1} = \{\mathbf{x} \in R^n : x_i = 0,\ \text{ for } \ i \in J^\circ(\mathbf{x}^k)\}$$

$k = k + 1$ and back to step 2.

Repetition of step 2 is simply an easy, economical recomputation of connection forces in the consecutively modified structure by producing the diad-modified inverse [4] of coefficient matrix $\mathbf{D}$, yielding, after the ith modification, for internal loads in the structure:

$$\mathbf{L}_i = \mathbf{L}_{i-1} - \mathbf{B}\mathbf{D}_{i-1}^{-1}\mathbf{B}_i^T(\mathbf{B}_i\mathbf{D}_{i-1}^{-1}\mathbf{B}_i^T + \Delta\mathbf{R}_i^{-1})^{-1}\mathbf{L}_{i-1,i}$$

where:

$\mathbf{L}_{i-1}$ internal loads in the structure after the i-1th modification;
$\mathbf{L}_{i-1,i}$ minormatrix of $\mathbf{L}_{i-1}$ (row vectors for elements modified i times);
$\mathbf{B}_i$ minormatrix of $\mathbf{B}$ after the ith modification
$\Delta\mathbf{R}_i$ flexibility matrix of the ith modification

and

$$\mathbf{D}_{i-1}^{-1} = \mathbf{D}_{i-2}^{-1} - \mathbf{D}_{i-2}^{-1}\mathbf{B}_{i-1}^T(\mathbf{B}_{i-1}\mathbf{D}_{i-2}^{-1}\mathbf{B}_{i-1}^T + \Delta\mathbf{R}_i^{-1})^{-1}\mathbf{B}_{i-1}\mathbf{D}_{i-2}^{-1}$$

Remind that under the actual limiting condition:

$$\Delta\mathbf{R}_i^{-1} = \mathbf{0}.$$

Convergence analysis

Any of points $\mathbf{x}^k$ is an admitted point. If condition in step 5 is met, then in fact, a solution results, namely Kuhn—Tucker theorems in step 2 are met. Provided condition in step 4 is failed, a finite number of repetitions of step 7 leads to a subspace where minimum is at $\mathbf{x}^k$. If then condition in step 5 is failed, the obtained y_{kj} is not yet a solution of the minimization problem; antigradient of the objective function defines a downward slope. Finite number of repetitions of step 7 yields another point y_{kj+1} that is a place for a minimum in the corresponding subspace. Function value tending to decrease along the set of points $y_{kj}, y_{kj+1}, \ldots,$ subspaces where these points define places of minima cannot be identical, involving different sets of subscripts to define them. There being a finite number of possible groupings of a finite number of subscripts, the algorithm has to end in a finite number of steps.

Now, the complementarity theorem has been utilized where conditions in step 5 were failed that would yield $\lambda_{j^c} < 0$. Now λ_{j^c} is deduced from the basis defining the system to be replaced by the corresponding x_{j^c}.

The procedure is simplified by the orthogonality of the limiting conditions, thus, if certain coordinates failed the condition of non-positivity, then in the other coordinates the projection on the corresponding coordinate axes of the point kept its sign, an easy way to obtain admitted points.

Analysis of structural members with nonlinear stiffness characteristics under general limiting conditions

In the case of an awkward transformation of the basic system or of the prevalence of limiting conditions of a form

$$\mathbf{B}\mathbf{x} + \mathrm{a} \leq \mathbf{0}$$

more general than is the condition of non-positivity, the essentials of the algorithm may subsist, only a direct variation of projecting to, and minimizing in, the subspace has to be applied, taking care of the integrity of already fulfilled conditions. The algorithm relying on the method of conjugated gradients may suit to solve the problem, namely it accomplishes minimization in the subspace in a finite number of steps. Along the conjugated directions the function value decreases, and so the generated algorithm is expected to remain finite.

Let us find the minimum of

$$W(x) = \frac{1}{2}\mathbf{x}^T \mathbf{D} \mathbf{x} + \mathbf{d}_0^T \mathbf{x} + \gamma$$

under limiting condition

$$\mathbf{Bx} + \mathrm{a} \leq \mathbf{0},$$

supposing that there exists a vector x_0 such that $\mathbf{B}\,\mathbf{x}_0 + \mathbf{a} < \mathbf{0}$.

Writing each row of the system of conditions:

$$\mathbf{b}_i\mathbf{x} + a_i = 0, \; i \in J.$$

Now, the Kuhn—Trucker conditions become:

$$\mathbf{Dx} + \mathbf{B}^T\mathbf{u} + \mathbf{d}_0 = \mathbf{0}$$

$$\mathbf{Bx} + \mathbf{a} = -\boldsymbol{\lambda} \leq \mathbf{0}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

$$\mathbf{u}^T(\mathbf{Bx} + \mathbf{a}) = \mathbf{u}^T\boldsymbol{\lambda} = \mathbf{0}$$

This latter is the condition of *complementarity*, to be met coordinate-wise.

Introducing notation:

$$J^\circ(x) = \{i : \mathbf{b}_i\mathbf{x} + a_i = 0, \; i \in J\}$$

$\mathbf{B}_{J^\circ}$ denoting the matrix with vectors $\mathbf{b}_i$ under $i \in J^\circ(x)$ as rows.

Projector matrix $\mathbf{P}_{J^\circ} = \mathbf{B}_{J^\circ}^T(\mathbf{B}_{J^\circ}\mathbf{B}_{J^\circ}^T)^{-1}\mathbf{B}_{J^\circ}$ projects to the subspace stretched by row vectors of matrix $\mathbf{B}_{J^\circ}$.

The minimization problem may be solved as:

1. $$k = 0$$

2. $$J_k^\circ = J^\circ(\mathbf{x}^k), \; \mathbf{P}_{J_k^\circ} = \mathbf{B}_{J_k^\circ}^T(\mathbf{B}_{J_k^\circ}\mathbf{B}_{J_k^\circ}^T)^{-1}\mathbf{B}_{J_k^\circ}$$

3. $$\text{If} \quad (\mathbf{E} - \mathbf{P}_{J_k^\circ})(\mathbf{Dx}^k + \mathbf{d}_0) = \mathbf{0}$$

step 4, otherwise step 7 follows.

4. $$\text{If} \quad \mathbf{u}_k = (\mathbf{B}_{J_k^\circ}\mathbf{B}_{J_k^\circ}^T)^{-1}\mathbf{B}_{J_k^\circ}(\mathbf{Dx}^k + \mathbf{d}_0) \leq \mathbf{0} \quad \forall_i \in J^\circ(\mathbf{x}^k)$$

then $\mathbf{x}^* = \mathbf{x}^k$ and the computation ends, otherwise

5. $$\tau_{k+1} = \{\mathbf{x} \in R^n : \mathbf{b}_i\mathbf{x} + a_i = 0, \; i \in \{i \in J^\circ(\mathbf{x}^k) : \mathbf{u}_i^k \leq 0\} = J^{k+1}\}$$

6. $$x_{k+1} = \underset{\tau_{k+1}}{\operatorname{argmin}} \; W(\mathbf{x}) \text{ and back to step 2.}$$

7. $$\tau_{k+1} = \{\mathbf{x} \in R^n : \mathbf{b}_i\mathbf{x} + a_i = 0, \; i \in J^\circ(x) = J^{k+1}\}$$

and back to step 6.

$-\mathbf{u}^k$ obtained in step 4 is the minimum norm solution of equation $\mathbf{D}\,\mathbf{x} + \mathbf{d}_0 + \mathbf{B}_{J_k^\circ}^T\mathbf{u} = \mathbf{0}$, namely

$$(\mathbf{B}_{J_k^\circ}\mathbf{B}_{J_k^\circ}^T)^{-1}\mathbf{B}_{J_k^\circ} = \mathbf{B}_{J_k}^+$$

is the pseudo-inverse of matrix $\mathbf{B}_{J_k}$.

Minimization in subspace τ_{k+1} in step 6 may be accomplished by the following conjugated gradient procedure:

1. $$j = 0$$

$$\mathbf{p}_1 = -(\mathbf{E} - \mathbf{P}_{J\bar{k}})(\mathbf{D}\mathbf{x}^{k,j} + \mathbf{d}_0)$$

2. $$J_{j+1} = \{i \in J^{k+1} : \mathbf{b}_i \mathbf{p}_{j+1} > 0\}$$

$$\bar{\alpha}_{j+1} = \min_{i \in I_{j+1}} \frac{a_i - \mathbf{b}_i \mathbf{x}^{k,j}}{\mathbf{b}_i \mathbf{p}_{j+1}}$$

$$\alpha_{j+1} = \min \left(- \frac{\mathbf{p}_{j+1}^T \mathbf{D} \mathbf{x}^{k,j}}{\mathbf{p}_{j+1}^T \mathbf{D} \mathbf{p}_{j+1}}, \ \bar{\alpha}_{j+1} \right)$$

3. $$\mathbf{x}^{k,j+1} = \mathbf{x}^{k,j} + \alpha_{j+1} \mathbf{p}_{j+1}$$

4. $$\mathbf{p}_{j+1} = -(\mathbf{E} - \mathbf{P}_{J\bar{k}})(\mathbf{D}\mathbf{x}^{k,j} + \mathbf{d}_0) + \frac{\|(\mathbf{E} - \mathbf{P}_{J\bar{k}})(\mathbf{D}\mathbf{x}^{k,j} + \mathbf{d}_0)\|^2}{\|(\mathbf{E} - \mathbf{P}_{J\bar{k}})(\mathbf{D}\mathbf{x}^{k,j-1} + \mathbf{d}_0)\|^2} \mathbf{p}_j$$

Selection of the step length according to step 2 of the procedure provides for obtaining the minimum point under the limiting condition along the given direction p_{j+1} in a finite number of steps [5].

From the minimization problem under general limiting conditions also the case of simpler limiting conditions may be deduced, simplifying also the procedure.

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