NUMERICAL METHOD FOR THE STUDY OF THE PLANAR UNSTEADY-STATE FLOW OF COMPRESSIBLE VISCOUS LIQUIDS

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The immense development of electronic computers in the past quarter of a century permitted to realize the elaboration of numerical processes requiring a vast volume of computation work for solving flow problems in the field of hydrodynamics, considered so far as unmanageable, — reflecting more or less correctly the involved problems — irrespective of the a priori approximative character of these processes.

For studying the problems of the flow of non-viscous compressible liquids, a high number of various systems of equations were developed, reproducing the studied phenomenon with different degrees of accuracy. A very rich collection of the elaborated processes is found in (1).

Part of these processes (1), (2) apply the following systems of equations for studying the flow of compressible liquids:

For the momentum variations

$$\frac{\partial M}{\partial t} + \nabla \cdot (M\overline{w}) = -\frac{\partial p}{\partial x}$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\overline{w}) = -\frac{\partial p}{\partial y}$$

For the continuity

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho\overline{w}) = 0 .$$

For the variation of the intrinsic energy

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon\overline{w}) = -p \nabla \cdot \overline{w}$$

For the equation of state

$$p = p(\varepsilon, \varrho)$$

where

$M = \varrho u$, $N = \varrho v$ — unit volume liquid pulses in directions x and y resp.;
$\overline{W} = u\overline{i} + v\overline{j}$ — velocity of the liquid in directions $x(u)$ and $y(v)$ resp.;
ϱ — density;
p — pressure;
ε — intrinsic energy in unit volume of liquid.

In the present study the intrinsic energy of liquids is neglected, as it was prepared mainly for studying the flow of the liquids, but the terms related to the viscosity are taken into account.

Be μ the friction coefficient of the liquid, then the frictional components of the stress tensor in the liquid in planar flow are:

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x} + \frac{2}{3}\mu(\nabla \cdot \overline{w})$$

$$\tau_{yy} = -2\mu \frac{\partial v}{\partial y} + \frac{2}{3}\mu(\nabla \cdot \overline{w})$$

$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Let us for now these quantities into the vector quantities:

$$\overline{\tau}_x = \tau_{xx} \cdot \overline{i} + \tau_{xy} \cdot \overline{j}$$
$$\overline{\tau}_y = \tau_{xy} \cdot \overline{i} + \tau_{yy} \cdot \overline{j}$$

Introducing the divergence of these quantities into the equation of pulse variation:

$$\frac{\partial M}{\partial t} + \nabla \cdot (M\overline{w}) = -\frac{\partial p}{\partial x} - \nabla \cdot (\overline{\tau}_x)$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\overline{w}) = -\frac{\partial p}{\partial y} - \nabla \cdot (\overline{\tau}_y)$$

Derivates make it clear that the equations describe indeed the flow of compressible viscous liquids.

Both motion equations are completed by the continuity equation;

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \overline{w}) = 0$$

and the equation of state:

$$p = p(\varrho)$$

The seemingly enforced inclusion of the conception of divergence in the system of equations relies on formulae by F. N. Noh (1) for the determination of the partial derivates and the divergence, which very well suit numerical purposes.

The basic idea of the name is as follows:

Assuming the function 'f' of the rim ∂R lying in the plane x, y and interpreted in the range R to possess a sufficient number of derivates, then a point may be stated to exist in the range $R(x_0, y_0)$,

where $$\frac{\partial f(x_0, y_0)}{\partial x} = \oint_{\partial R} f \cdot dy / \oint_{\partial R} x \cdot dy$$

and a point (x_1, y_1),

where $$\frac{\partial f(x_1, y_1)}{\partial y} = -\oint_{\partial R} f \cdot dy / \oint_{\partial R} x \cdot dy$$

From both formulae it may be deduced that both derivates may be substituted, with a very good approximation, by the following relationships applied to the j^{th} compartment (Fig. 1)

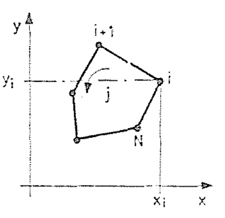

Fig. 1

I. $$\left(\frac{\Delta f}{\Delta x}\right)_j = \frac{\sum_{i=1}^{N} (f_{i+1} + f_i) \cdot (y_{i+1} - y_i)}{\sum_{i=1}^{N} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)}$$

II. $$\left(\frac{\Delta f}{\Delta y}\right)_j = -\frac{\sum_{i=1}^{N} (f_{i+1} + f_i) \cdot (x_{i+1} - x_i)}{\sum_{i=1}^{N} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)}$$

and with $\bar{w} = u \cdot \bar{i} + v \cdot \bar{j}$, the divergence of quantity $(f\bar{w})$ is

III. $$\nabla \cdot (f\bar{w})_j = \frac{\sum_{i=1}^{N} \left\{ \left[(fu)_{i+1} + (fu)_i\right] \cdot (y_{i+1} - y_i) - \left[(fv)_{i+1} + (fv)_i\right] \cdot (x_{i+1} - x_i) \right\}}{\sum_{i=1}^{N} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)}$$

Derivation formulae I, II and III are crucial for the process entering all the approximation formulae. The elaborated program, operates with compartments of three and four corner-points, but for the sake of simplicity only the four point compartments are considered in the following. If we assume that at the n^{th} step all the data are available, then the difference equation of the pulse variations after time interval Δt may be written as:

$$M_{k,l}^{n+1} = M_{k,l}^{n} - \Delta t \cdot \left\{ [\nabla \cdot (M\overline{w})]_{k,l}^{n} + \left(\frac{\Delta p}{\Delta x}\right)_{k,l}^{n} + [\nabla \cdot \overline{\tau}_x]_{k,l}^{n} \right\}$$

$$N_{k,l}^{n+1} = N_{k,l}^{n} - \Delta t \cdot \left\{ [\nabla \cdot (N\overline{w})]_{k,l}^{n} + \left(\frac{\Delta p}{\Delta y}\right)_{k,l}^{n} + [\nabla \cdot \overline{\tau}_y]_{k,l}^{n} \right\}$$

During the processes the pressure in the individual compartments is regarded as constant, so partial derivatives $\partial p/\partial x$ and $\partial p/\partial y$ at point (k, l) may be replaced by:

$$\left(\frac{\Delta p}{\Delta x}\right)_{k,l}^{n} = \frac{p_1^n + p_2^n - p_3^n - p_4^n}{2(x_1 - x_4)}$$

$$\left(\frac{\Delta p}{\Delta x}\right)_{k,l}^{n} = \frac{p_2^n + p_3^n - p_1^n - p_4^n}{2(y_2 - y_1)}$$

Here x_1, x_4, y_1 and y_2 are coordinates of the compartment centre. This means essentially that the derivatives at point (k,l) are expressed by the contents of the four surrounding compartments.

The divergence of the momentum may be obtained for the general range with the help of expression III. For the case of Fig. 2

$$[\nabla \cdot (M\overline{w})]_{k,l}^{n} = \frac{(Mu\,\Delta y)_{k+1/2,l}^{n} + (Mv\,\Delta x)_{k,l+1/2}^{n} - (Mu\,\Delta y)_{k-1/2,l}^{n} - (Mv\,\Delta x)_{k,l-1/2}^{n}}{(x_1 - x_4)\cdot(y_2 - y_1)}$$

The place x of the interpretation of quantities with subscript 1/2 is shown in Fig. 3.

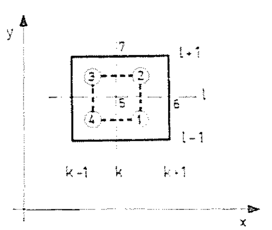

Fig. 2

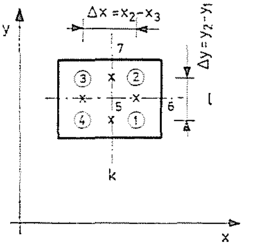

Fig. 3

With the designations

$$u^n_{k+1/2,\,l} = \frac{u^n_5 + u^n_6}{2}$$

$$v^n_{k,\,l+1/2} = \frac{v^n_5 + v^n_7}{2}$$

we may write

$$(Mu\,\Delta y)^n_{k+1/2,\,l} = \Delta y \cdot u^n_{k+1/2,\,l} \cdot \begin{cases} M^n_5 & \text{if} \quad u^n_{k+1/2,\,l} \geqq 0 \\ M^n_6 & \text{if} \quad u^n_{k+1/2,\,l} < 0 \end{cases}$$

$$(Mv\,\Delta x)^n_{k,\,l+1/2} = \Delta x \cdot v^n_{k,\,l+1/2} \cdot \begin{cases} M^n_5 & \text{if} \quad v^n_{k,\,l+1/2} \geqq 0 \\ M^n_7 & \text{if} \quad v^n_{k,\,l+1/2} < 0 \end{cases}$$

etc. The divergence of the momentum N in direction y is calculated in the same way.

For calculating the expressions $\nabla \cdot (\overline{\tau}_x)$ and $W \cdot (\tau_y)$, first, components τ_{xx}, τ_{yy} and τ_{xy} have to be determined. As velocities u^n and v^n at corner points of each compartment are known, their partial derivates, interpreted for the compartment may be calculated with the help of expressions I and II. τ_{xy}, τ_{xx} and τ_{yy} may be calculated from the derivates as described above. In their knowledge divergences $\nabla \cdot (\tau_x)^n$ and $\nabla \cdot (\overline{\tau}_y)^n$, valid at point (k, l), are interpreted as the divergences of the rectangles laid on the gravity centres of the four grid elements limiting point (k, l), i.e., essentially they are calculated similarly to the pulse divergences.

Now the new pulses $M^{n+1}_{k,l}$ and $N^{n+1}_{k,l}$ can be calculated from the components.

The density valid at point (k, l) is calculated as the mean density of the four compartments surrounding the point, by the following expression

$$\varrho^n_{k,l} = \frac{1}{4}\left(\varrho^n_{k+1/2,\,l+1/2} + \varrho^n_{k-1/2,\,l+1/2} + \varrho^n_{k-1/2,\,l-1/2} + \varrho^n_{k+1/2,\,l-1/2}\right).$$

The new velocities are obtained from the pulse variations and the density as

$$u_{k,l}^{n+1} = \frac{M_{k,l}^{n+1}}{\varrho_{k,l}^{n}}; \qquad v_{k,l}^{n+1} = \frac{N_{k,l}^{n+1}}{\varrho_{k,l}^{n}}$$

The density prevailing in the grid centre is obtained from the continuity equation as:

$$\varrho_{k+1/2,\,l+1/2}^{n+1} = \varrho_{k+1/2,\,l+1/2}^{n} - \Delta t \cdot \left[\nabla \cdot (\varrho \overline{w})\right]_{k+1/2,\,l+1/2}^{n}.$$

On the basis of the expression of divergence applied to the rectangular ranges:

$$\left[\nabla (\varrho \cdot \overline{w})\right]_{k+1/2,l+1/2}^{n} = \frac{(\varrho u \Delta y)_{k+1/2,l+1/2}^{n} + (\varrho v \Delta x)_{k+1/2,l+1}^{n} - (\varrho u \Delta y)_{k,l+1/2}^{n} - (\varrho v \Delta x)_{k+1/2,l}^{n}}{(x_{k+1} - x_k) \cdot (y_{l+1} - y_l)}.$$

Introducing designations

$$u_{k+1,\,l+1/2}^{n+1} = (u_{k+1,\,l+1}^{n+1} + u_{k+1,\,l}^{n+1})/2.$$

$$v_{k+1/2,\,l}^{n+1} = (v_{k+1,l+1}^{n+1} + v_{k,l+1}^{n+1})/2.$$

we may write

$$(\varrho u \Delta y)_{k+1,\,l+1/2}^{n} = \Delta y \cdot u_{k+1,\,l+1/2}^{n+1} \cdot \begin{cases} \varrho_{k+1/2,\,l+1/2}^{n} & \text{if} \quad u_{k+1,\,l+1/2}^{n+1} \geqq 0 \\ \varrho_{k+3/2,\,l+1/2}^{n} & \text{if} \quad u_{k+1,\,l+1/2}^{n+1} < 0 \end{cases}$$

and

$$(\varrho v \Delta x)_{k+1/2,\,l+1}^{n} = \Delta x \cdot v_{k+1/2,\,l+1}^{n+1} \cdot \begin{cases} \varrho_{k+1/2,\,l+1/2}^{n} & \text{if} \quad v_{k+1/2,\,l+1}^{n+1} \geqq 0 \\ \varrho_{k+1/2,\,l+3/2}^{n} & \text{if} \quad v_{k+1/2,\,l+1}^{n+1} < 0. \end{cases}$$

i.e. the pre-existing densities and the new velocity are used for determining the divergence.

After the determination of the new densities the new pressures at the compartment centre can be calculated with the help of the equation of state

$$p_{k+1/2,\,l+1/2}^{n+1} = p(\varrho_{k+1/2,\,l+1/2}^{n+1}).$$

Now, by the end of the step interval Δt all the data required for starting the next step interval are available.

Our tests performed so far show that the behaviour of the described process is — with a correctly chosen step interval Δt — utmost stable. Accordind to our experiences it is sufficient to have a Δt value shorter than the time required for the wave propagation to attain one third of the narrowest grid dimension, at the sound velocity in the liquid.

The presented example shows the case where the velocity of the liquid rises in the x-direction instantaneously to 30 m/sec along the rim of the range surrounding the vane submerged in the liquid; the result shown in the figure represents the state after the 500-th step (Fig. 4).

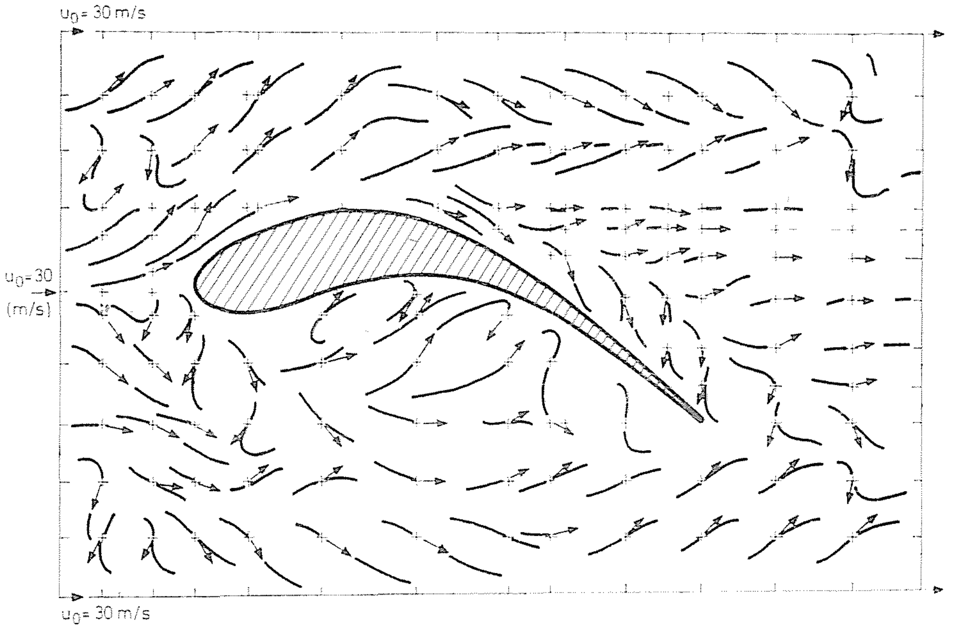

Fig. 4

The vortex developing under the belly of the blade is shown excellently. The track of the particles, i.e. the streamline drawn in the figure is given by the curves fitted to speed directions in each grid point.

Summary

This paper is concerned — by way of computer — with comparatively novel approach to the solution of the governing differential equations of physics for viscous compressible fluid motion in plane, at unsteady state flow. The paper describes the essential features of the new method.

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