PRELIMINARY APPROXIMATE ANALYSIS OF BENDING STRESSES IN LATTICE-LIKE VEHICLE STRUCTURES

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Introduction

Statically indeterminate structures can only be designed by iteration, since determination of structural sizes requires the knowledge of stresses, depending, in turn, on cross-sectional data of structural members. Preliminary design is, therefore, advisably made as an approximation. In this stage, a simple mathematical model is recommended, pointing out essential features of the examined phenomenon of structure.

One of the most frequent models of utility vehicle structures is the lattice. Lattices suit modelling e.g. vehicle frames, autobus floor frames, or in certain cases even the complete vehicle can be considered as a lattice.

Several, exact and approximate, methods have been developed for the calculation of lattices, based either on the force or the displacement method [1].

Design methods have been applied for the analysis of stresses first in bridges, later also in vehicles [2].

In the following, analysis of a lattice with four longitudinal beams (e.g. a vehicle frame) will be presented, applying a mathematical method offering a fast approximation of stresses to ease preliminary decisions and to evaluate the effect of necessary further modifications.

The problem will be solved by the force method, taking, in general, internal work due to bending alone into consideration.

1. *Development of the model*

Vehicle structures are mostly symmetric. Bus bodies have, however, no symmetry axis in the strict sense of the word. A strict symmetry is offset, in addition to the asymmetry due to minor, negligible details, by the one side door cuts (Fig. 1). The stress distribution in a structure of disturbed symmetry may much differ from that in symmetric structures, imposing them to be analysed, though direct determination in asymmetric structures is rather cumbersome.

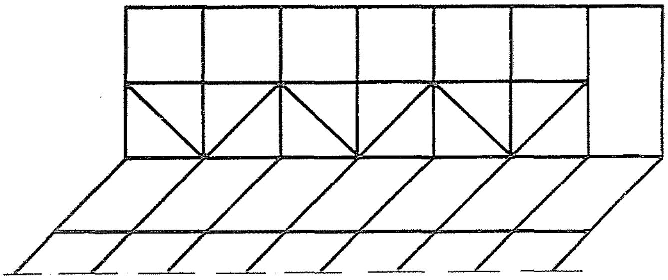


Fig. 1

Vehicle structure in Fig. 1 can also be modelled as a lattice (Fig. 2). The lattice model is assumed to consist of members all able to absorb bending stresses and intersecting members to be joined by hinges. As a further simplification, in the model the longitudinal beams are considered to be of constant stiffness throughout, and cross-beams to be of identical stiffness (regular lattice). Again, symmetry about the longitudinal axis is assumed.

The possibility of accessory consideration of asymmetry — e.g. door cuts — initially neglected to simplify conditions will be detailed later.

No difficulties arise from the determination of lattice member stiffness for both inner longitudinal beams and cross-beams, especially for framed vehicles. In autobuses and railway coaches sidewall stiffness is also affected by the window field. Rather than to abruptly increase the number of unknowns, its accessory effect is advisably assessed by approximation. The equivalent sidewall stiffness yields, at the same time, stiffness of the outer longitudinal beam of the lattice.

Determination of the equivalent sidewall stiffness advisably starts from the identify of displacements between defined points of the real structure and the substituting model. Remind, however, that the equivalent stiffness depends, in addition to the structure design, also on the load distribution. In the subsequent considerations, the load distribution will be assumed — according to earlier observations — as close to the real one as possible.

The grid structure under the window of the real sidewall in Fig. 3 can be considered as a deep-web beam where shear is absorbed by grid bars, and

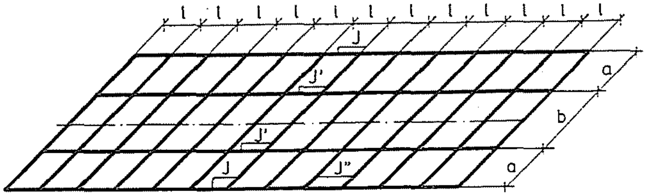


Fig. 2

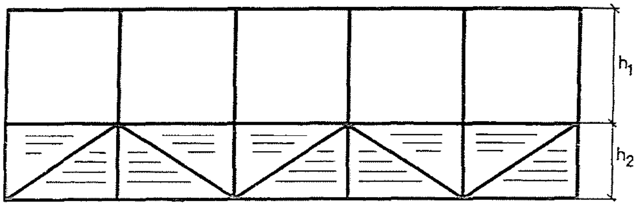


Fig. 3

bending by flange bars. Omitting the deformation caused by shear forces (grid bar length changes), inertia J_1 of the deep-web beam can be determined, permitting to replace the real sidewall by the simplified model in Fig. 4 for further calculations.

Assuming the top flange and the window post to be hinged, the model is more elastic than the real structure. The model can be refined by the "modified Fabry method" [3], taking also the bending stiffness of the top flange into consideration (reduced window post stiffness). In the following, inertia of the window post will be assumed to involve this correction.

The sidewall in Fig. 4 is replaced in calculations by a single beam, of an inertia depending on the window post inertia J_2, the parapet inertia J_1, and the cross-sectional are A of the top flange.

Let us pick out e.g. a two-field part of the structure (Fig. 5). Calculation of the equivalent inertia starts from the identity between deflections of defined points B. In either case, the structure is loaded by the same force F_0. In calculating the internal works let us take bending, and for the top flange,

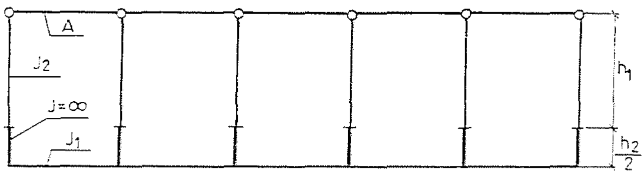


Fig. 4

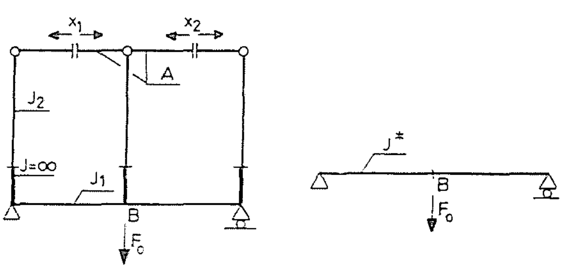


Fig. 5

the normal stresses into consideration. The equivalent inertia obviously depends on the number of involved frames, too.

For the sidewall construction actually applied (in buses) the order of magnitude is closely approximated by:

$$J^* = \left\{\left[\frac{n}{2}\right] \cdot c + 1\right\} J_1$$

where n — number of fields,

$\left[\frac{n}{2}\right]$ — highest integer number comprised in $n/2$;

c — computed value (≈ 0.05).

Under different stiffness conditions, of course, also the c value differs. Significant differences may arise for railway coaches where window posts are stiffer than are those of buses.

In practice, $J^* \approx (1.1 \text{ to } 1.3)\, J_1$. Also the effect of door cuts to affect the stiffness has to be examined. In these places, bending and shear stiffnesses are much modified. Also here, the equivalent inertia can be deduced from the identity of displacements.

Provided no high shear forces arise in the vicinity of door cuts of the real structure, neither shear in the substituting model will be reckoned with (Fig. 6.) This omission is valid to mid-car-body doors.

With respect to actual stiffness conditions:

$$J^{**} \approx (0.15 - 0.2)\, J^*.$$

In the case of high shear forces also the effect of shear is to be involved in calculating the displacements, as seen on the substituting model in Fig. 7.

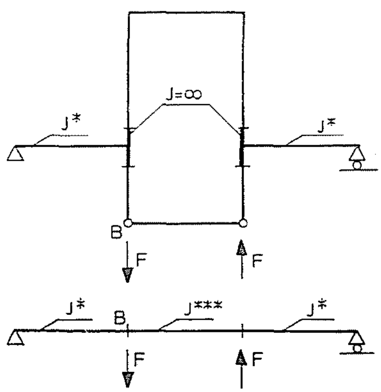


Fig. 6

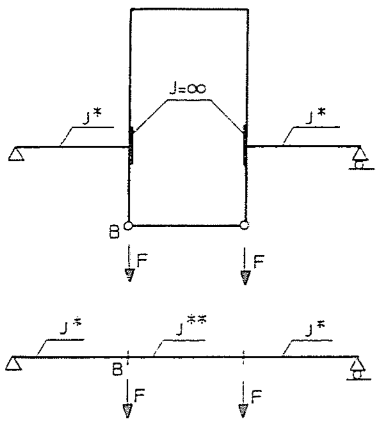


Fig. 7

The equivalent inertia becomes:

$$J^{***} \approx (0.02 - 0.03)\,J^{*}.$$

a substitution valid for car-body-end doors.

2. *Regular lattice with stiff cross-beams*

Let us consider the moment developing in the outer longitudinal beam (sidewall) at cuts above the cross-beams due to unknown internal forces in the regular lattice with stiff cross-beams as seen in Fig. 8.

The compatibility equation is of the form

$$\mathbf{DX} - \mathbf{d} = \mathbf{0} \tag{1}$$

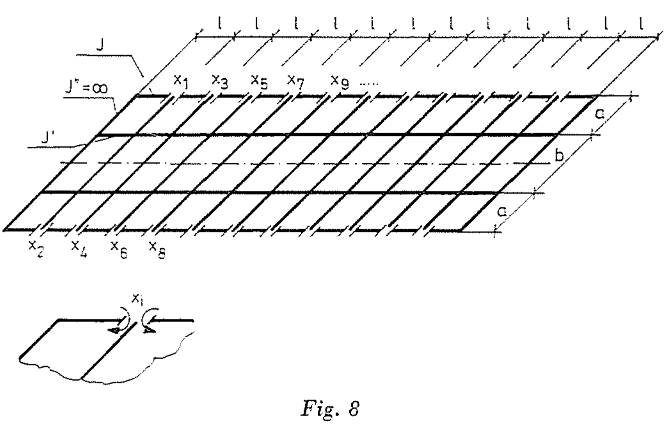


Fig. 8

where

D matrix of unit factors;
X matrix of unknowns;
d matrix of load factors.

A possible means to determine the unknown internal moments by producing the inverse of **D** is:

$$\mathbf{X} = -\mathbf{D}^{-1}\mathbf{d}. \tag{2}$$

The stresses of lattice are obtained according to the principle of superposition:

$$M = M_0 + \Sigma X_i m_i \tag{3}$$

where

M_0 basic structure stress due to outer loads;
X_i internal moment at the i-th cut;
m_i bending stress due to unit moment pair acting at the i-th cut.

The usual production of the inverse is, however, rather cumbersome, and unreliable because of the superposition of rounding-off errors. So the coefficient matrix is decomposed into matrices with inverses producible in closed form.

For the structure in Fig. 8, the coefficient matrix

$$\underset{(n\times n)}{\mathbf{D}} = \begin{bmatrix} \alpha & \gamma & \beta & \delta & & & & & & \\ \gamma & \alpha & \delta & \beta & & & & & & \\ \beta & \delta & \alpha & \gamma & \beta & \delta & & & & \\ \delta & \beta & \gamma & \alpha & \delta & \beta & & & & \\ & & \beta & \delta & \alpha & \gamma & \beta & \delta & & \\ & & \delta & \beta & \gamma & \alpha & \delta & \beta & & \\ & & & & \beta & \delta & \alpha & \gamma & \beta & \delta \\ & & & & \delta & \beta & \gamma & \alpha & \delta & \beta \\ & & & & & & & & \ddots & \\ & & & & & & & & & \ddots \end{bmatrix}$$

where

n number of redundancies;
E modulus of elasticity;

$$\alpha = \frac{2l}{3E}\left[\frac{1}{J} + \frac{\left(1+\frac{a}{b}\right)^2}{J'} + \frac{\left(\frac{a}{b}\right)^2}{J'}\right]$$

$$\gamma = -\frac{4l}{3E}\,\frac{\left(\frac{a}{b}\right)\left(1+\frac{a}{b}\right)}{J'}$$

$$\beta = \alpha/4 \qquad \delta = \gamma/4.$$

Taking the relationship between unit factors into consideration:

$$\mathbf{D} = \begin{bmatrix} 4\beta & 4\delta & \beta & \delta & & & & & & \\ 4\delta & 4\beta & \delta & \beta & & & & & & \\ \beta & \delta & 4\beta & 4\delta & \beta & \delta & & & & \\ \delta & \beta & 4\delta & 4\beta & \delta & \beta & & & & \\ & & \beta & \delta & 4\beta & 4\delta & \beta & \delta & & \\ & & \delta & \beta & 4\delta & 4\beta & \delta & \beta & & \\ & & & & \beta & \delta & 4\beta & 4\delta & \beta & \delta \\ & & & & \delta & \beta & 4\delta & 4\beta & \delta & \beta \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & \cdot\;\cdot & \cdot \\ & & & & & & & & & \cdot\;\cdot\;\cdot \end{bmatrix} = \begin{bmatrix} \beta & \delta \\ \delta & \beta \end{bmatrix} \otimes \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 & 1 \\ & & & \cdot\;\cdot & \cdot \\ & & & & \cdot\;\cdot\;\cdot \end{bmatrix} = \mathbf{K} \otimes \mathbf{L} \tag{4}$$

According to the theorem for the inverse of direct matrix products:

$$\mathbf{D}^{-1} = (\mathbf{K} \otimes \mathbf{L})^{-1} = \mathbf{K}^{-1} \otimes \mathbf{L}^{-1}. \tag{5}$$

Rather than to directly invert matrix $\mathbf{D}$ direct product of two easily produced inverses has to be applied. $\mathbf{L}$ is a special continuant matrix, with an inverse producible in closed form:

$$\mathbf{L}^{-1} = [r_{ij}]$$

and

$$r_{ij} = \begin{cases} (-1)^{i+j} \dfrac{\operatorname{sh}(i\Theta)}{\operatorname{sh}\Theta} \dfrac{\operatorname{sh}[(n+1-j)\Theta]}{\operatorname{sh}[(n+1)\Theta]} & \text{if } i \leq j \\[2ex] (-1)^{i+j} \dfrac{\operatorname{sh}(j\Theta)}{\operatorname{sh}\Theta} \dfrac{\operatorname{sh}[(n+1-i)\Theta]}{\operatorname{sh}[(n+1)\Theta]} & \text{if } i \geq j \end{cases} \tag{6}$$

where

$$\Theta = \ln(2 + \sqrt{3}).$$

and the inverse of matrix $\mathbf{K}$:

$$\mathbf{K}^{-1} = \frac{1}{\beta^2 - \delta^2} \begin{bmatrix} \beta & -\delta \\ -\delta & \beta \end{bmatrix} \tag{7}$$

3. *Regular lattice with elastic cross-beams*

Coefficient matrix of the compatibility equation will keep its band structure even for elastic cross-beams, however, at an increased bandwidth:

$$\underset{(n \times n)}{\mathbf{D}} = \begin{bmatrix} \alpha' & \gamma' & \beta' & \delta' & \varepsilon' & \eta' & & & & & & & & \\ \gamma' & \alpha' & \delta' & \beta' & \eta' & \varepsilon' & & & & & & & & \\ \beta' & \delta' & \alpha' & \gamma' & \beta' & \delta' & \varepsilon' & \eta' & & & & & & \\ \delta' & \beta' & \gamma' & \alpha' & \delta' & \beta' & \eta' & \varepsilon' & & & & & & \\ \varepsilon' & \eta' & \beta' & \delta' & \alpha' & \gamma' & \beta' & \delta' & \varepsilon' & \eta' & & & & \\ \eta' & \varepsilon' & \delta' & \beta' & \gamma' & \alpha' & \delta' & \beta' & \eta' & \varepsilon' & & & & \\ & & \varepsilon' & \eta' & \beta' & \delta' & \alpha' & \gamma' & \beta' & \delta' & \varepsilon' & \eta' & & \\ & & \eta' & \varepsilon' & \delta' & \beta' & \gamma' & \alpha' & \delta' & \beta' & \eta' & \varepsilon' & & \\ & & & & \varepsilon' & \eta' & \beta' & \delta' & \alpha' & \gamma' & \beta' & \delta' & \varepsilon' & \eta' \\ & & & & \eta' & \varepsilon' & \delta' & \beta' & \gamma' & \alpha' & \delta' & \beta' & \eta' & \varepsilon' \\ & & & & & & & & & \cdot\;\cdot & \cdot & & & \\ & & & & & & & & & & \cdot\;\cdot\;\cdot & & & \\ & & & & & & & & & & & \cdot\;\cdot\;\cdot & & \end{bmatrix}$$

where

$$\alpha' = \alpha + \frac{2}{J'' E}\left(\frac{a}{l}\right)^2 (a+b)$$

$$\beta' = \beta - \frac{4}{3J'' E}\left(\frac{a}{l}\right)^2 (a+b)$$

$$\gamma' = \gamma + \frac{1}{J'' E}\left(\frac{a}{l}\right)^2 b$$

$$\delta' = \delta - \frac{2}{3J'' E}\left(\frac{a}{l}\right)^2 b$$

$$\varepsilon' = \frac{1}{3J'' E}\left(\frac{a}{l}\right)^2 (a+b)$$

$$\eta' = \frac{1}{6J'' E}\left(\frac{a}{l}\right)^2 b.$$

Elements of matrix $\mathbf{D}$ have been determined from the model in Fig. 9.

The coefficient matrix being a hypermatrix of secondary cyclic blocks, that is, the modal matrix is the same for any block, namely the involutory matrix

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{8}$$

similarity (actually, involutory) transformation by matrix $\mathbf{T}$ in every block leads to hypermatrix of secondary diagonal matrices, equivalent to the decomposition of the system into two independent part-systems.

Hence:

$$\mathbf{C}(\mathbf{T}\otimes\mathbf{E})\,\mathbf{D}(\mathbf{T}\otimes\mathbf{E})^*\mathbf{C}^* =$$

$$= \left[\begin{array}{cccccc|cccccc} A_1 & B_1 & C_1 & & & & & & & & & \\ B_1 & A_1 & B_1 & C_1 & & & & & & & & \\ C_1 & B_1 & A_1 & B_1 & C_1 & & & & & & & \\ & C_1 & B_1 & A_1 & B_1 & C_1 & & & & & & \\ & & & \cdot & \cdot & \cdot & & & & & & \\ & & & \cdot & \cdot & \cdot & & & & & & \\ \hline & & & & & & A_2 & B_2 & C_2 & & & \\ & & & & & & B_2 & A_2 & B_2 & C_2 & & \\ & & & & & & C_2 & B_2 & A_2 & B_2 & C_2 & \\ & & & & & & & C_2 & B_2 & A_2 & B_2 & C_2 \\ & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & \cdot & \cdot & \cdot \end{array}\right] = \mathbf{K}_1'\oplus\mathbf{K}_2' \tag{9}$$

where $\mathbf{C}$ is a permuting matrix rearranging elements of a hypermatrix of secondary blocks to produce a hypermatrix partitioned to four blocks.

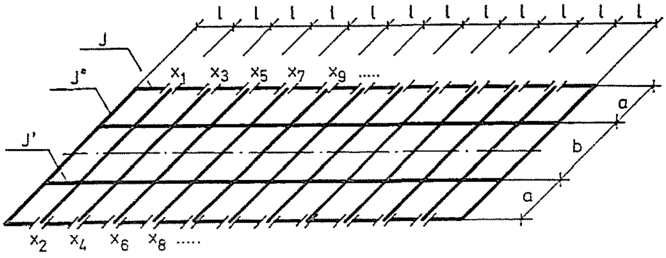


Fig. 9

Introducing terms

$$\mu_1 = \frac{l^3 J''\left[\frac{1}{J} + \frac{1}{J'}\right]}{a^2(3b + 2a)} \tag{10}$$

and

$$\mu_2 = \frac{l^3 J''\left[\frac{1}{J} + \left(\frac{b + 2a}{b}\right)^2 \frac{1}{J'}\right]}{a^2(b + 2a)} \tag{11}$$

yields:

$$\mathbf{K}_1' = C_1 \begin{bmatrix} 6 + 4\mu_1 & \mu_1 - 4 & 1 & & & & \\ \mu_1 - 4 & 6 + 4\mu_1 & \mu_1 - 4 & 1 & & & \\ 1 & \mu_1 - 4 & 6 + 4\mu_1 & \mu_1 - 4 & 1 & & \\ & 1 & \mu_1 - 4 & 6 + 4\mu_1 & \mu_1 - 4 & 1 & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & \ddots \end{bmatrix} \tag{12}$$

and

$$\mathbf{K}_2' = C_2 \begin{bmatrix} 6 + 4\mu_2 & \mu_2 - 4 & 1 & & & & \\ \mu_2 - 4 & 6 + 4\mu_2 & \mu_2 - 4 & 1 & & & \\ 1 & \mu_2 - 4 & 6 + 4\mu_2 & \mu_2 - 4 & 1 & & \\ & 1 & \mu_2 - 4 & 6 + 4\mu_2 & \mu_2 - 4 & 1 & \\ & & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & \end{bmatrix} \tag{13}$$

The obtained matrices $\mathbf{K}_1'$ and $\mathbf{K}_2'$ can only be decomposed in form

$$\mathbf{K}' = N(\mathbf{M}_1 \mathbf{M}_2 + \mathbf{M}_3) \tag{14}$$

Although inverses of continuant matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are known, but matrix $\mathbf{M}_3$ leads to a further two-diacid modification.

It is recognized to obtain — by modifying the model — a matrix structure that can be decomposed into the product of two matrices of known inverses.

Let the original structure be completed by sections of elements of infinite stiffness according to Fig. 10 on the left and the right side.

Transformation according to the above of the compatibility equation of the resulting model yields:

$$(\mathbf{K}_1 \oplus \mathbf{K}_2)\,\mathbf{u} = \mathbf{f} \tag{15}$$

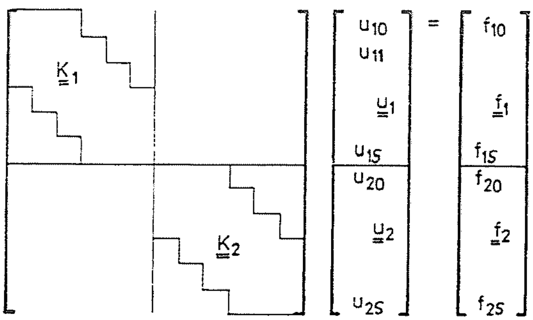
where

$$\mathbf{K}_1 = C_1 \begin{bmatrix} 5+4\mu_1 & \mu_1-4 & 1 & & & & \\ \mu_1-4 & 6+4\mu_1 & \mu_1-4 & 1 & & & \\ 1 & \mu_1-4 & 6+4\mu_1 & \mu_1-4 & 1 & & \\ & & & & & & \\ & & 1 & \mu_1-4 & 6+4\mu_1 & \mu_1-4 \\ & & & 1 & \mu_1-4 & 5+4\mu_1 \end{bmatrix} \tag{16}$$

and

$$\mathbf{K}_2 = C_2 \begin{bmatrix} 5+4\mu_2 & \mu_2-4 & 1 & & & & \\ \mu_2-4 & 6+4\mu_2 & \mu_2-4 & 1 & & & \\ 1 & \mu_2-4 & 6+4\mu_2 & \mu_2-4 & 1 & & \\ & & & & & & \\ & & 1 & \mu_2-4 & 6+4\mu_2 & \mu_2-4 \\ & & & 1 & \mu_2-4 & 5+4\mu_2 \end{bmatrix} \tag{17}$$

Thus, the original compatibility equation is decomposed into two independent (although slightly increased) matrix equations.



Matrices being of identical built-up, both equations are similarly solved, therefore, solution of a single matrix equation will be presented.

Let us consider equation

$$\mathbf{K}_1\,\mathbf{u}_1 = \mathbf{f}_1 \tag{19}$$

solved in the form:

$$\mathbf{u}_1 = \mathbf{K}_1^{-1}\,\mathbf{f}_1 \tag{20}$$

Since in developing the new model internal moments have been assumed also where they were zero, in solving the equation system first, the f_{10} and f_1 values have to be determined under conditions

$$\begin{aligned} u_{10} &= 0 \\ u_{1s} &= 0 \end{aligned} \tag{21}$$

Introducing notations

$$p = \frac{(\mu_1 - 4) - \sqrt{\mu_1^2 - 24\,\mu_1}}{2} \tag{22}$$

and

$$q = \frac{(\mu_1 - 4) + \sqrt{\mu_1^2 - 24\,\mu_1}}{2} \tag{23}$$

as well as matrices

$$\mathbf{P} = \begin{bmatrix} p & 1 & & & & & & \\ 1 & p & 1 & & & & & \\ & 1 & p & 1 & & & & \\ & & 1 & p & 1 & & & \\ & & & 1 & p & 1 & & \\ & & & & 1 & p & 1 & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot \; \cdot \\ & & & & & & & \cdot \; \cdot \; \cdot \end{bmatrix} \tag{24}$$

$$\mathbf{Q} = \begin{bmatrix} q & 1 & & & & & & \\ 1 & q & 1 & & & & & \\ & 1 & q & 1 & & & & \\ & & 1 & q & 1 & & & \\ & & & 1 & q & 1 & & \\ & & & & 1 & q & 1 & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot \; \cdot \\ & & & & & & & \cdot \; \cdot \; \cdot \end{bmatrix} \tag{25}$$

the coefficient matrix becomes:

$$\mathbf{K}_1 = C_1 \mathbf{P} \mathbf{Q} \tag{26}$$

and its inverse:

$$\mathbf{K}_1^{-1} = \frac{1}{C_1} \mathbf{Q}^{-1} \mathbf{P}^{-1}. \tag{27}$$

Inverse of matrices **Q** and **P** can be determined in closed form by means of second-kind Tchebyshev polynomials of degree n [4].

4. *Relationship between inverses of coefficient matrices applied in the analysis of regular lattices with stiff and elastic cross-beams*

Involutory transformation of the coefficient matrix of the regular lattice with elastic cross-beam, and arranging by a permuting matrix results in a hypermatrix partitioned into four blocks. Hypermatrices in the main diagonal are of similar structure, so that the deduction leading to the relationship above will only be presented for one of them.

After transformation and permutation (Eq. 16):

$$\mathbf{K}_1 = C_1 \begin{bmatrix} 5+4\mu_1 & \mu_1-4 & 1 & & & & & & \\ \mu_1-4 & 6+4\mu_1 & \mu_1-4 & 1 & & & & & \\ 1 & \mu_1-4 & 6+4\mu_1 & \mu_1-4 & 1 & & & & \\ & 1 & \mu_1-4 & 6+\mu_1 4 & \mu_1-4 & 1 & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & \mu_1-4 & 6+4\mu_1 & \mu_1-4 \\ & & & & & & 1 & \mu_1-4 & 5+4\mu_1 \end{bmatrix}$$

Be

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \ \cdot \\ & & & & \cdot & \cdot \ \cdot \end{bmatrix} \tag{28}$$

Hence

$$\begin{aligned} \mathbf{K}_1 &= C_1 [\mu_1(6\mathbf{E} - \mathbf{A}) + \mathbf{A}^2] = C_1 \mu_1 \left[[6\mathbf{E} - \mathbf{A} + \frac{1}{\mu_1} \mathbf{A}^2 \right] = \\ &= C_1 \mu_1 (6\mathbf{E} - \mathbf{A}) \left[\mathbf{E} + \frac{1}{\mu_1} (6\mathbf{E} - \mathbf{A})^{-1} \mathbf{A}^2 \right]. \end{aligned} \tag{29}$$

and its inverse:

$$\mathbf{K}_1^{-1} = \frac{1}{C_1 \mu_1} \left[\mathbf{E} + \frac{1}{\mu_1} (6\mathbf{E} - \mathbf{A})^{-1} \mathbf{A}^2 \right]^{-1} (6\mathbf{E} - \mathbf{A})^{-1}. \tag{30}$$

In producing $\mathbf{K}_1^{-1}$, the second factor in the right-hand side of Eq. (30) can be series expanded, and since according to practical experience, it suffices to reckon with the first two terms of series expansion, it is:

$$\left[\mathbf{E} + \frac{1}{\mu_1} (6\mathbf{E} - \mathbf{A})^{-1} \mathbf{A}^2 \right]^{-1} \approx \mathbf{E} - \frac{1}{\mu_1} (6\mathbf{E} - \mathbf{A})^{-1} \mathbf{A}^2 \tag{31}$$

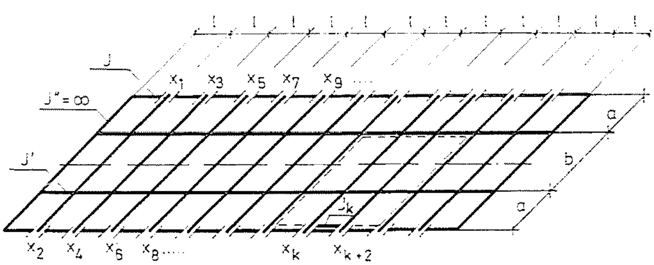


Fig. 10

hence

$$\mathbf{K}_1^{-1} \approx \frac{1}{C_1 \mu_1} (6\mathbf{E} - \mathbf{A})^{-1} - \frac{1}{C_1 \mu_1^2} [(6\mathbf{E} - \mathbf{A})^{-1} \mathbf{A}]^2. \tag{32}$$

The first term yields the inverse of the coefficient matrix of the lattice with an infinitely stiff cross-beam, while the second term is proportional to the effect of elastic cross-beams to modify the stress distribution.

This inverse in rather simple to produce namely also $(6\mathbf{E} - \mathbf{A})^{-1}$ can be produced in closed form.

5. *Reckoning with the stiffness change of structural members in inverting the coefficient matrix of the compatibility equation*

Be the coefficient matrix of the compatibility equation produced in the form

$$\mathbf{D} = \mathbf{B}^* \mathbf{R}\mathbf{B} \tag{33}$$

where

$\mathbf{B}$ matrix of stresses from unit load pairs;

$\mathbf{R}$ flexibility matrix of the structure.

Change of the stiffness of the i-th member of the structure alters the block (i, i) of the original flexibility matrix. Considering the modified flexibility matrix as sum of two matrices:

$$\mathbf{R}_m = \mathbf{R} + \mathbf{R}_M \tag{34}$$

the coefficient matrix becomes:

$$\mathbf{D}_M = \mathbf{B}^* \mathbf{R}_m \mathbf{B} = \mathbf{B}^*(\mathbf{R} + \mathbf{R}_M)\mathbf{B} = \mathbf{B}^*\mathbf{R}\mathbf{B} + \mathbf{B}^* \mathbf{R}_M \mathbf{B} = \mathbf{D} + \mathbf{M} \tag{35}$$

Let us consider a minimum diadic decomposition of M:

$$\mathbf{M} = (\mathbf{B}^* \mathbf{R}_M) \mathbf{B}. \tag{36}$$

Thus, the inverse of the modified coefficient matrix:

$$\mathbf{D}_M^{-1} = (\mathbf{D} + \mathbf{M})^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{B}^* \mathbf{R}_M(\mathbf{E} + \mathbf{B}\mathbf{D}^{-1} \mathbf{B}^* \mathbf{R}_M)^{-1} \mathbf{B}\mathbf{D}^{-1}. \tag{37}$$

Diagonal hypermatrix $\mathbf{R}_M$ differing from the zero matrix by a single block, it is sufficient to calculate with a minor matrix each of matrices $\mathbf{B}, \mathbf{B}^*$, $\mathbf{R}_M$, i.e. matrices $\mathbf{B}_m$, $\mathbf{B}_m^*$, $\mathbf{R}_{Mm}$.

Possible transformations lead to the general formula:

$$\mathbf{D}_M^{-1} = \mathbf{D}^{-1} - (\mathbf{B}_m \mathbf{D}^{-1})^* (\mathbf{R}_{Mm}^{-1} + \mathbf{B}_m \mathbf{D}^{-1} \mathbf{B}_m^*)^{-1} \mathbf{B}_m \mathbf{D}^{-1} \tag{38}$$

to be further simplified by taking pecularities of the model into consideration. The further simplification is possible by the development of stresses due to unit equilibrium moments acting at cut joints only in a defined range of, rather than over the complete, structure.

Let us consider the modification of the regular lattice with stiff cross-beams.

Fig. 11 shows only unit loads X_k and X_{k+2} to produce stresses in the section to be modified. Hence:

$$\mathbf{B}_m = \begin{array}{c} \quad\quad x_k \quad\quad x_{k+2} \\ \begin{bmatrix} \ldots . 1 & . \; 0 \ldots \\ \ldots . 0 & . \; 1 \ldots \end{bmatrix} \end{array}. \tag{39}$$

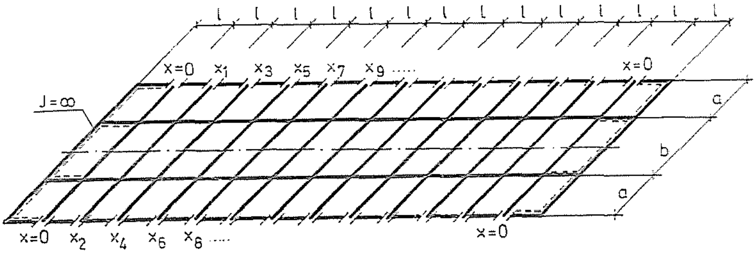


Fig. 11

Denoting $\widetilde{\mathbf{B}}_m$ a minor matrix of $\mathbf{B}_m$:

$$\widetilde{\mathbf{B}}_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{40}$$

and be

$$\mathbf{D}_m^{-1} = \left[\frac{D_{k,i}^{-1}}{D_{k+2,i}^{-1}} i = 1, 2, \ldots, n \text{———} \right] \tag{41}$$

and

$$\widetilde{\mathbf{D}}_m^{-1} = \left[\begin{array}{c|c} D_{k,k}^{-1} & D_{k,k+2}^{-1} \\ \hline D_{k+2,k}^{-1} & D_{k+2,\,k+2}^{-1} \end{array} \right] \tag{42}$$

minor matrices of $\mathbf{D}^{-1}$.

Determination of the inverse flexibility matrix $\mathbf{R}_{Mm}^{-1}$ starts from connecting another beam of inertia J_k to the original beam of inertia J. Then, assuming the stress to vary at most linearly along the tested section,

$$\mathbf{R}_{Mm}^{-1} = -\frac{2(J + J_k)\,E}{l} \frac{J}{J_k} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{43}$$

Thereby the inverse of the modified coefficient matrix becomes:

$$\mathbf{D}_M^{-1} = \mathbf{D}^{-1} - (\mathbf{D}_m^{-1})^* (\mathbf{R}_{Mm}^{-1} + \widetilde{\mathbf{D}}^{-1})^{-1} \mathbf{D}_m^{-1} \tag{44}$$

where the inverse of a single (2×2) matrix has to be calculated, other matrices being at disposal.

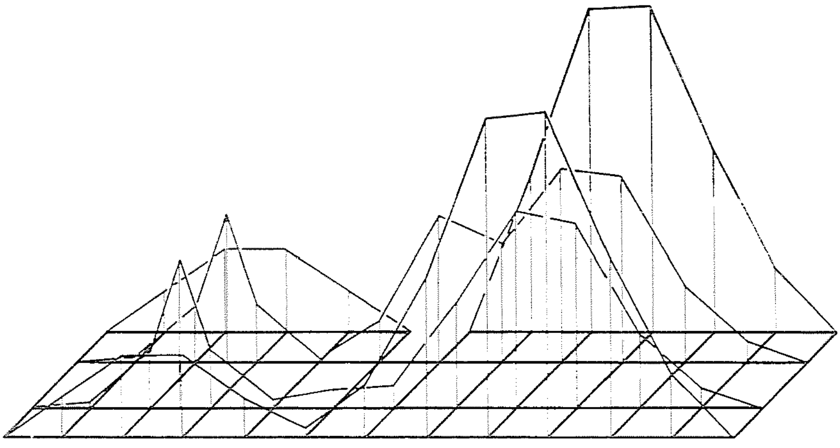


Fig. 12. No door

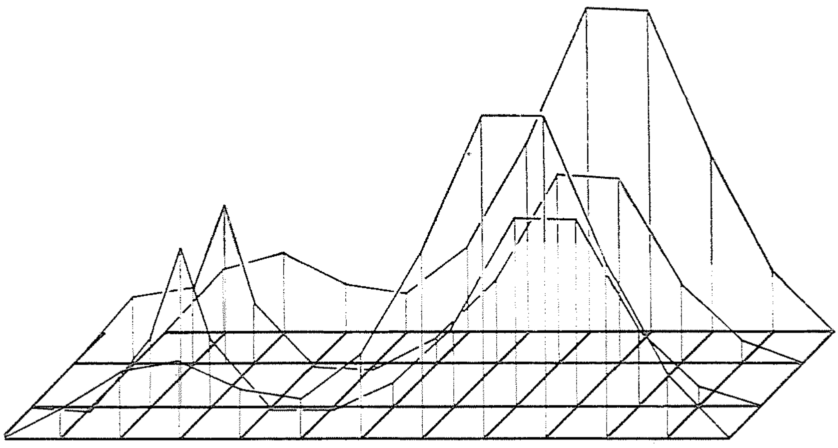


Fig. 13. With a forward door

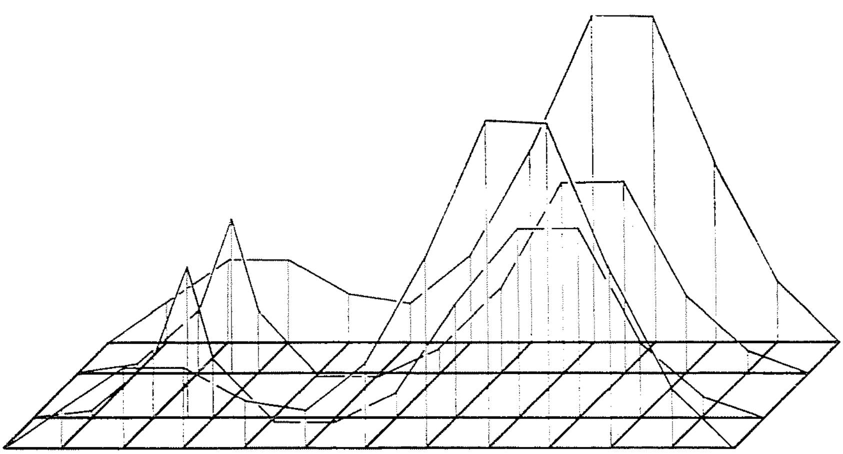


Fig. 14. With a middle door

Of course, the general modification formula deduced from purely mathematical considerations equals the relationship by Argyris and Kelsey [5], relying on thermal displacements, as well as the modification method deduced by Nándori from the general coupling procedure [6, 7].

The effect of stiffness change — due to stresses indicated for longitudinal beams alone — have been plotted in Figs 12, 13, 14. Throughout the calculations member stiffnesses — except the modified parts — have been considered as constant.

Summary

The presented calculation method lends itself to the preliminary, approximate stress analysis of lattices with four principal beams, stiff or elastic cross-beams, exempt from torsion. The solution methods based on the force method permit to produce coefficient matrix inverses in closed form, and to take various structural modifications into consideration by inverting only small (2×2) new matrices.

Inverses possible in closed form reduce numerical errors resulting from computer inversions, permitting to solve the problem by small-capacity desk computers or even minicomputers.

References

1. PALOTÁS, L.: Analysis of Lattices* Közlekedési Kiadó, Budapest, 1953.
2. SAMU, B.—MICHELBERGER, P.: Annäherungsrechnung der Beanspruchung von Omnibustragwerken. Periodica Polytechnica, Budapest, 1974.
3. BARÁNSZKY—JÓB, I.: Manual of Railway Cars* Műszaki Könyvkiadó, Budapest, 1967.
4. RÓZSA, P.: Linear Algebra and Application.* Műszaki Könyvkiadó, Budapest, 1976.
5. ARGYRIS, I. H.—KELSEY, S.: Modern Fuselage Analysis and the Elastic Aircraft. Butterworths. London, 1963.
6. NÁNDORI, E.: Car Bodies with Closed Superstructure.* Doctor Techn. Thesis, Budapest, 1972.
7. NÁNDORI, E.: Effect of Stiffness Variation of Under- and Overdimensioned Structural Members of Hyperstatic Vehicle Structures with Multiple Redundancies.* Járművek, Mezőgazdasági Gépek, 21. No. 10. p. 366, 1974.

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