UTILIZATION OF QUASI-SYMMETRY IN THE ANALYSIS OF LINEAR MECHANICAL SYSTEMS

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1. Introduction

Speaking of a structure, symmetry is mostly meant as geometrical symmetry. Its simplest case is symmetry about a plane where at least one plane exists, such that the tested object reflected by it is mapped onto itself. In the following, symmetry is always understood as simple symmetry about a plane, no utilization of eventual other (geometry) symmetries is attempted.

In mechanical analyses, symmetries of rigidity, damping, mass distribution characteristics, or of loads are considered as extensions of geometrical symmetry. These latter will be arbitrarily termed *external* symmetries, as an unambiguous distinction from internal symmetries of linear mechanics, formulated in exchangeability theorems, relying on deeper physical relationships.

In the occurence of external symmetry in these terms (to be called further on: symmetry), on the one hand, the computation work may be reduced by halving or still more subdividing the tested structure into independent substructures, and on the other hand, numerical computations are more reliable and easier to evaluate.

Often, the structure is only "nearly" symmetric, hence quasi-symmetric, because of usual small-area disturbances of the overall symmetry. Familiar problems of this kind are — in the case of vehicles — frame symmetry troubles due to one-sided doors of autobuses, or to the location of mechanical equipment in motor coaches.

Making use of advantages residing in symmetry will be shown to be exempt from problems, therefore the natural demand arises to handle quasi-symmetric cases in a way to manage the greatest part of advantages fully exploitable in perfect symmetry cases. Structural analyses may benefit from a procedure relying on the connection principle [1, 2], providing for the quoted advantages also in the quasi-symmetric case by aptly selecting the load vector so that in any step of computation one has to do with a symmetric structure. For the dynamic analysis of quasi-symmetric cases there is, however, no exact method available at present.

A method will be suggested for the dynamic analysis of quasi-symmetric cases such that if a latent but close symmetry exists, it helps to recognize it. permitting analysis of the structure as a symmetric one. For an effectively disturbed symmetry, the suggested method facilitates selection of the mechanically dominant symmetry, to underlie a fairly close approximation.

2. Analysis of symmetrical systems by utilizing the symmetry

This study will refer to linear systems or their matrix representations for some coordinate selection.

2.1 Static analyses

In static analysis, a mechanical system may be described e.g. by the stiffness matrix $\mathbf{S}$ of order n, and by generalized displacements $\mathbf{q}$ due to generalized load $\mathbf{f}$. Hence

$$\mathbf{S}\mathbf{q} = \mathbf{f} \tag{1}$$

The system is assumed to be symmetric, there is a matrix $\mathbf{Q}$ — usually with several alternatives — permitting transfer to coordinates $\mathbf{y}$ reflecting symmetry conditions of the structure through transformation

$$\mathbf{q} = \mathbf{Q}\mathbf{y} \tag{2}$$

replacing, for instance, pair of the rotation-translation coordinates disrupting the symmetry by a pair of translation coordinates. Since coordinates $\mathbf{y}$ already reflect the symmetry conditions of the structure, an order may be found where every coordinate joins its counterpart in symmetry — provided n is even. A vector with this feature may formally be obtained from

$$\mathbf{y} = \mathbf{P}\mathbf{x} \tag{3}$$

where $\mathbf{P}$ is an aptly selected permuting matrix of order n. Coordinate transformations (2) and (3) lead to

$$(\mathbf{P}^* \mathbf{Q}^* \mathbf{S}\mathbf{Q}\mathbf{P})\mathbf{x} = \mathbf{P}^* \mathbf{Q}^* \mathbf{f} \tag{4}$$

of a form suitable to be joined by a non-singular matrix $\mathbf{D}$ constructed as described in [3] such that transformation

$$\mathbf{D}^*(\mathbf{P}^* \mathbf{Q}^* \mathbf{S}\mathbf{Q}\mathbf{P})\mathbf{D}\,\mathbf{z} = \mathbf{D}^*(\mathbf{P}^* \mathbf{Q}^* \mathbf{f}) \tag{5}$$

results in two independent systems.

Introducing notation $\mathbf{T} = \mathbf{Q}\mathbf{P}\mathbf{D}$ yields

$$\mathbf{T}^* \mathbf{S}\mathbf{T}\mathbf{z} = \mathbf{T}^* \mathbf{f} \tag{6}$$

containing matrix $\mathbf{T}^*\mathbf{ST}$, a diagonal hypermatrix with two blocks of exactly halved size if n is even.

If n is odd, then one coordinate has no symmetric counterpart (it being in the symmetry plane): initially the pertaining equation can be considered as to belong to any subsystem.

Provided the system has several external symmetries — subsystem symmetries — the obtained, half-size problems may be further reduced.

The same procedure is deduced in [3] by the flexibility rather than by the stiffness matrix, with no essential difference.

The actual selection relies on the compatibility with the usual formulation of dynamic problems.

2.2 Dynamic analyses

The structural problem under 2.1 the dynamic problems of the form

$$\mathbb{M}\,\ddot{\mathbf{q}} + \mathbb{K}\,\dot{\mathbf{q}} + \mathbb{S}\,\mathbf{q} = \mathbf{f}(t) \tag{7}$$

can be assigned to, where the here introduced symmetric matrices $\mathbb{M}$ and $\mathbb{K}$ are inertia and damping characteristics of the system, resp., and t is time.

Applying the same procedure as before,

$$\mathbb{T}^*\,\mathbb{M}\mathbb{T}\ddot{\mathbf{z}} + \mathbb{T}^*\,\mathbb{K}\mathbb{T}\dot{\mathbf{z}} + \mathbb{T}^*\,\mathbb{S}\mathbb{T}\mathbf{z} = \mathbb{T}^*\,\mathbf{f}(t) \tag{8}$$

For a system identically involving symmetries of rigidity, damping and inertia, matrix $\mathbb{T}$ selected as described above, primarily depending on the order of coordinates $\mathbf{y}$ causes also Eq. (8) to decompose.

Let us mention that the somewhat mechanical establishment of matrix $\mathbb{D}$ in [3] may be replaced by another method. For the treatment of vehicles, in the chosen coordinate system, matrix $\mathbb{M}$ is often diagonal, $\mathbb{K}$ and $\mathbb{S}$ are of similar construction, since dampings are modelled as parallel connected to elastic elements. No matter that the damping matrix often contains more of zero elements than the stiffness one, that is, from the aspect of procedure, occurrence of cases $s_{ij} \neq 0$, $k_{ij} = 0$ for some subscript pairs i, j is irrelevant.

These items may be utilized in constructing transformation matrices depending on the model parameters.

It is worth mentioning that to transformation matrices or transformed equations mostly interesting physical interpretations may be assigned, since models of lower degrees of freedom of the type (8) may hint to peculiarities of the original problem, likely to remain hidden in the original form (7). Remind that, provided the component transformation matrix (8) is non-singular, then this is a similarity transformation with respect to the algebraic eigenvalue problem belonging to the dynamic problem, that is, eigenvalues do not change in transformation, only that coordinates of the eigenvalues are obtained in the new system.

E x a m p l e s:

As an illustration of those said above, let us consider the symmetric vibrating system with four degrees of freedom seen in Fig. 1.

The displacement equation in the system of coordinates q_i:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & \frac{m_s a^2 + J_s}{4a^2} & \frac{m_s a^2 - J_s}{4a^2} \\ 0 & 0 & \frac{m_s a^2 - J_s}{4a^2} & \frac{m_s a^2 + J_s}{4a^2} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{bmatrix} +$$

$$+ \begin{bmatrix} k_1 + k_3 & 0 & -k_3 & 0 \\ 0 & k_1 + k_3 & 0 & -k_3 \\ -k_3 & 0 & k_3 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} +$$

$$+ \begin{bmatrix} s_1 + s_3 & 0 & -s_3 & 0 \\ 0 & s_1 + s_3 & 0 & -s_3 \\ -s_3 & 0 & s_3 & 0 \\ 0 & -s_3 & 0 & s_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}.$$

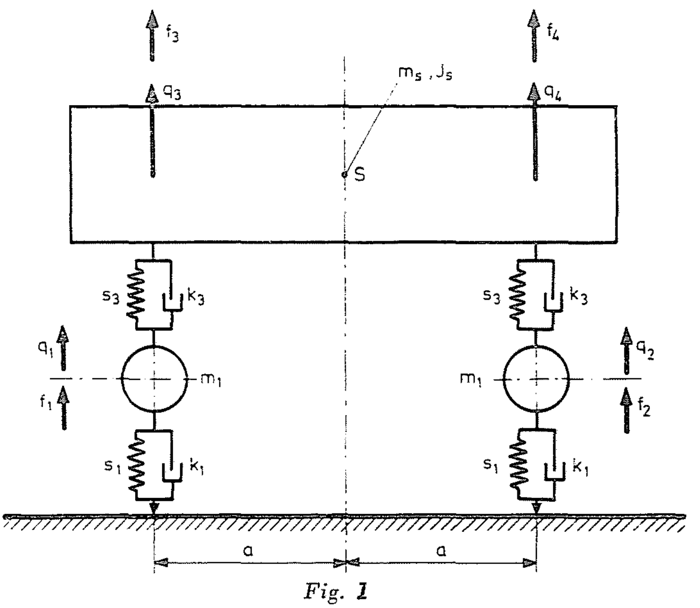

Fig. 1

In the selected coordinate system, the coefficient matrices of the displacement equation do not decompose into independent blocks, however many zeros they contain.

a) Utilization of symmetry by transformation according to [3].

The coordinates being symmetric and needing no change of sequence,

$$\mathbf{Q} = \mathbf{P} = \mathbf{E} \text{ (unit matrix)}$$

$$\mathbf{y} = \mathbf{x} = \mathbf{q}$$

and $\mathbf{T} = \mathbf{D}$, where

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{i. e.} \quad \mathbf{D}^* = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

According to Eq. (8):

$$\begin{bmatrix} 2m & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 \\ 0 & 0 & J_s/a^2 & 0 \\ 0 & 0 & 0 & m_s \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \\ \ddot{z}_4 \end{bmatrix} +$$

$$+ \begin{bmatrix} 2(k_1 + k_3) & 0 & -2k_3 & 0 \\ 0 & 2(k_1 + k_3) & 0 & -2k_3 \\ -2k_3 & 0 & 2k_3 & 0 \\ 0 & -2k_3 & 0 & 2k_3 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} +$$

$$+ \begin{bmatrix} 2(s_1 + s_3) & 0 & -2s_3 & 0 \\ 0 & 2(s_1 + s_3) & 0 & -2s_3 \\ -2s_3 & 0 & 2s_3 & 0 \\ 0 & -2s_3 & 0 & 2s_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} f_1 - f_2 \\ f_1 + f_2 \\ f_3 - f_4 \\ f_3 + f_4 \end{bmatrix}$$

a decomposing system where equations containing unknowns of odd subscript do not contain such with even subscript, and vice versa.

b) Another separation possibility

Be

$$\mathbf{T}=\begin{bmatrix} 1 & -a & 0 & 0 \\ 1 & a & 0 & 0 \\ \hdashline 1 & -a & 1 & -a \\ 1 & a & 1 & a \end{bmatrix} \quad \text{and} \quad \mathbf{T}^{*}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ -a & a & -a & a \\ \hdashline 0 & 0 & 1 & 1 \\ 0 & 0 & -a & a \end{bmatrix}.$$

Performing multiplications specified in (8):

$$\begin{bmatrix} 2m+m_s & 0 & m_s & 0 \\ 0 & 2a^2m+J_s & 0 & J_s \\ m_s & 0 & m_s & 0 \\ 0 & J_s & 0 & J_s \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \\ \ddot{z}_4 \end{bmatrix} +$$

$$+\begin{bmatrix} 2k_1 & 0 & 0 & 0 \\ 0 & 2a^2k_1 & 0 & 0 \\ 0 & 0 & 2k_3 & 0 \\ 0 & 0 & 0 & 2a^2k_3 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} +$$

$$+\begin{bmatrix} 2s_1 & 0 & 0 & 0 \\ 0 & 2a^2s_1 & 0 & 0 \\ 0 & 0 & 2s_3 & 0 \\ 0 & 0 & 0 & 2a^2s_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} f_1+f_2+f_3+f_4 \\ a(f_2-f_1+f_4-f_3) \\ f_3+f_4 \\ a(f_4-f_3) \end{bmatrix}$$

again a decomposing system. but while subsystems transformed in the former way are stiffness and damping coupled in themselves. the latter method yields subsystems acceleration-coupled in themselves but remaining independent of each other.

3. Analysis of quasi-symmetric structures

Solution of the symmetric case was seen earlier to require in final account the performance of a congruent transformation where the transformation matrix can be algorithmized even in the most general case. Existence of the mentioned conditions is sufficient for the utilizability of symmetry.

3.1 Static analysis in quasi-symmetric cases

In a quasi-symmetric case, the exact solution is produced in two steps:

— Arbitrary restitution of symmetry by structural intervention; solution of the symmetrical problem according to 2.1.

— Symmetric and antisymmetric modification of the artificially symmetrized structure (coupling problem) so as to recover the original structure at last.

The essential of the procedure is seen in Fig. 2.

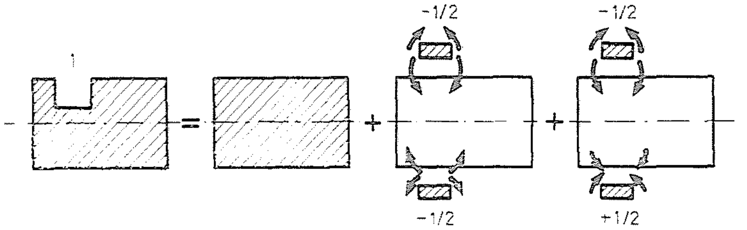

Fig. 2

Every step of the procedure has been developed and is actually achievable [4, 2]. The computation method can be considered as convenient selection of load vectors applied on the symmetric structure, since the symmetry disturbance of the given structure under a given outer load may be interpreted as if the symmetric structure replacing the quasi-symmetric one were acted upon, in addition to the original load, by another, aptly selected external equilibrium load.

3.2 Dynamic analysis

The fundamental difficulty of utilizing quasi-symmetry in dynamic cases is due — to our knowledge — to the missing generalization development of the coupling problem [5] for general cases. As a result, ulterior correction of modifications to restore the symmetry cannot be performed.

Earlier, the static analysis of quasi-symmetric structures consisted in an approximation omitting the step of modification. Essentially the same method will be followed now, except that as far as possible the slightest modification likely to result in a symmetric system will be sought for.

There are several ways of symmetrization:

a) The considered problem is subjected to physical-type modifications such that the equation system in form (7) or (8) becomes a decomposing one. In the simplest case springs, masses, dampers are omitted or assumed, type or number of connections is modified etc. In general, this is a rough intervention causing the model to less truly simulate the examined system, to specialize, to reduce the number of adjustable parameters. the obtained results are misleading without correction. Even essential pecularities of the examined system were found to vanish in this kind of symmetrization.

b) Rather than to find the basis vectors, congruent coordinate transformations, simple to construct, will be introduced where though the equations do not decompose but there are few coupling terms, and of low value compared to the elements in the main diagonal of the corresponding matrix. Omission of the remaining coupling terms obviously causes the system to decompose. It is equally expedient to but slightly modify the equations before transformation e.g. by aptly completing them so that after transformation again a decomposing system arises.

The latter procedure means a smaller intervention in the system than that under a) since number and range of the parameters remain the same as in the original problem, the essential of the modification being the slightly biased reckoning with some interactions of the system along some coordinates.

4. Estimation of approximation errors

Errors of the numerical solution values arise from two sources: finite arithmetics, and biassing by the model. Influence of finite arithmetics will not be considered in detail, only mentioned that symmetric and minor problems can mostly be solved more efficiently and accurately than the original quasi-symmetric problem. One may wonder if a slightly inaccurate numerical solution of a large-size, quasi-symmetric, theoretically exacter model is the more advantageous, or a relatively more accurate and more reliable numerical solution of smaller, theoretically less exact models — or of a symmetric model of the original size but of lower accuracy. Thus, errors may be expected to offset each other.

A natural demand is, however, to measure or estimate the errors produced either numerically or by model biassing, and their impact on the solution. Static computations involve the solution of linear algebraic equation systems, so that the theorems below [6] suit error calculation, although the actual error is sometimes overestimated.

Theorem A: If $\|\mathbf{A}-\mathbf{C}\| < \frac{1}{\|\mathbf{A}^{-1}\|}$ then also $\mathbf{C}^{-1}$ exists such that

$$\|\mathbf{A}^{-1}-\mathbf{C}^{-1}\| \leq \frac{\|\mathbf{A}-\mathbf{C}\|\ \|\mathbf{A}^{-1}\|^2}{1-\|\mathbf{A}-\mathbf{C}\|\ \|\mathbf{A}^{-1}\|}\ .$$

Theorem B: Assume the equation system $\mathbf{Ax} = \mathbf{f}$ to be unambiguously solvable for $\mathbf{x}$. For $\|\mathbf{A}-\mathbf{C}\| < \frac{1}{\|\mathbf{A}^{-1}\|}$ then for an arbitrary $\mathbf{g}$, also equation system $\mathbf{Cy} = \mathbf{g}$ can be unambiguously solved (see the previous theorem) and the solutions differ by:

$$\|\mathbf{x}-\mathbf{y}\| \leq \frac{\|\mathbf{A}^{-1}\| \, (\|\mathbf{A}-\mathbf{C}\| \, \|\mathbf{x}\| + \|\mathbf{g}-\mathbf{f}\|)}{1-\|\mathbf{A}-\mathbf{C}\| \, \|\mathbf{A}^{-1}\|} .$$

In the following, matrices **C** and **A** will be considered as exact matrix of the quasi-symmetric structural problem, and as coefficient matrix of an aptly selected approximate symmetric problem, respectively.

Dynamic analyses involve the solution of both linear algebraic equations (steady vibrations), and linear, differential equations with constant coefficients (transient processes). Again relying on [6], a theorem will be presented for the most delicate part of this latter problem, error assessment of the eigenvalue problem solution. It should be mentioned a priori that it is practically not sharp enough for a narrow delimitation of the real error.

c) *A. Ostrowski's theorem.* Let us assume eigenvalues of matrices **A** and **C**, both of order n, to be $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ respectively. Be

$$M = \max_{1 \leq i, j \leq n} \{ |a_{ij}|, |c_{ij}| \}$$

$$\delta = \frac{1}{nM} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} - c_{ij}| .$$

Now, for an arbitrary eigenvalue λ_k of matrix **A**, at least one eigenvalue μ_m of **C** can be found, such that:

$$|\lambda_k - \mu_m| \leq (n+2) \, M \, \delta^{1/n} \qquad 1 \leq k, \; m \leq n .$$

4.1 *Example of applying error assessment in a structural problem*

Assume of a quasi-symmetric static problem

$$\mathbf{S} \, \mathbf{q} = \mathbf{f}$$

has led, using earlier described transformations, to an equation system of the form:

$$\begin{bmatrix} 8 & 2 & \varepsilon & \\ 2 & 8 & & \\ \gamma & & 4 & 1 \\ & & 1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{gathered} \varepsilon = \gamma \\ \text{(in compliance with the} \\ \text{theorems of exchange-} \\ \text{ability)} \end{gathered}$$

For $\varepsilon = \gamma = 0$, coupling of main diagonal blocks vanishes, and the problem requires the solution of only two, half-size equation systems, as against the original, completely coherent system. Thereby a possible symmetric approximation of the original quasi-symmetric problem has been obtained; let us try to assess how close the solutions of both problems are.

To be able to apply the presented theorems to this aim, let us consider the coefficient matrix pertaining to the original (detailed) quasi-symmetric case as **C**, the pertaining solution vector as **y**, while matrix of the decomposing problem resulting from the selection $\varepsilon = \gamma = 0$ be **A**, and the pertaining solution vector be **x**, right-hand sides being common: **f** = **g**. This selection is justified by the doubtless existence and easy calculability of the inverted of **A** (exactly this was the aim of the neglect: to obtain an approximate problem easier to handle than the original one), **A** being a diagonal hypermatrix of non-singular blocks.

Numerical calculations apply the Euclidean norm $\| \;\|_2$ although theorems *A* and *B* are valid in any compatible vector — matrix norm [7].

Partial results needed for evaluating theorem *B*:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{f} = \frac{1}{30}[4, -1, -2, 8]$$

$$\|\mathbf{A}^{-1}\| = 0.43; \quad \|\mathbf{x}\| = 0.31;$$

$$\|\mathbf{A} - \mathbf{C}\| = + \sqrt{\varepsilon^2 + \gamma^2} \quad (\varepsilon = \gamma)$$

This error assessment can only be applied in the range $|\varepsilon| < 1.65$, inequality $+\sqrt{\varepsilon^2 + \gamma^2} < \dfrac{1}{0.43}$ being imperative. According to the theorem, the error limit is function of $|\varepsilon|$, some relevant numerical results are tabulated in Table 1.

Table 1:

$\|\varepsilon\| = \|\gamma\|$	$\|\mathbf{A} - \mathbf{C}\|$	$\|\mathbf{x} - \mathbf{y}\|$	$\|\mathbf{x} - \mathbf{y}\| / \|\mathbf{x}\|$
0.1	0.141	0.020	6.5%
0.5	0.707	0.135	43.7%
1.0	1.414	0.481	155.2%

As a conclusion, solution of the quasi-symmetric (exactly modelled) problem can safely be replaced, within engineering accuracy, by the solution of an artificially symmetrized (approximately modelled) problem, provided perturbation (symmetry disturbance) does not exceed $|\varepsilon| = |\gamma| = 0.1$.

By way of checking, also the exact solution of the quasi-symmetric problem has been determined, permitting to calculate the effective values in the two last columns of Table 1 (Table 2).

Confrontation of results in Tables 1 and 2 shows assessment according to theorem *B* though to much overestimate the effective deviation between vectors **x** and **y** but in the case $|\varepsilon| = |\gamma| \to 0$, the assessment accuracy rapidly converges, namely the assessment theorem reflects the continuous

Table 2:

$\|\varepsilon\| = \|\gamma\|$	$\|\|\mathbf{x}-\mathbf{y}\|\|$	$\|\|\mathbf{x}-\mathbf{y}\|\|/\|\|\mathbf{x}\|\|$
0.1	0.004	1.3%
0.5	0.011	3.5%
1.0	0.043	13.9%

dependence of the solutions from the coefficient matrix and the right-hand-side vector.

This example clearly illustrates the possibility, in case of a suitable organization of the static analysis of quasi-symmetric structures, of using "sharpening" error assessment formulae, fitting accuracy requirements of engineering practice, thus, likely to offer decision criteria for reducing the volume of calculations.

Let us notice that in the static case, exchangeability theorems specify existence of the symmetry of the stiffness matrix, hence of the relationship $\varepsilon = \gamma$. Decomposition depends on the existence of external symmetry: in the actual case it is confirmed by the value $\varepsilon = \gamma = 0$.

4.2 Example of applying error assessment for a dynamic problem

To the structural example above, dynamic problem, Eq. (7) can be assigned, considered, for the ease of treatment, to be of the form:

$$\mathbf{E}\ddot{\mathbf{q}} + \mathbf{S}\mathbf{q} = \mathbf{0} \tag{9}$$

Differential equation (9) may be considered to have a solution describing undamped free vibrations of the structure with the same stiffness matrix as in the former example, provided the mass matrix has unit elements. This is no essential restriction, since, obviously, the mass matrix is always symmetric, positive definite due to physical causes, thereby it can also be written in the form $\mathbf{LL}^*$, where $\mathbf{L}$ is a non-singular lower triangle matrix; making use of this decomposition, earlier symmetry conditions — if any — of matrix $\mathbf{S}$ can be maintained [8], while the mass matrix is transformed to unit matrix.

Similarly as in the static problem, also here it is attempted to decide, without solving the quasi-symmetric problem (matrix $\mathbf{C}$), whether results obtained from the solution of an approximate symmetric problem (matrix $\mathbf{A}$) can be accepted as sufficiently accurate solutions of the original problem.

To solve Eq. (9), knowledge of all eigenvalues of the algebraic eigenvalue problem

$$\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\omega^2 + \begin{bmatrix} 8 & 2 & \varepsilon & \\ 2 & 8 & & \\ \gamma & & 4 & 1 \\ & & 1 & 4 \end{bmatrix}\right)\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \mathbf{0} \tag{10}$$

is needed, therefore, it is attempted to apply theorem *C*) and to use eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of matrix **A** belonging to the symmetric case $\varepsilon = \gamma = 0$ to approximate eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$ of the original matrix **C**, and to assess the error.

4.2.1 Assessment relying on theorem *C*

Matrix **A** being a diagonal hypermatrix, its eigenvalues are readily obtained:

$$\lambda_1 = 3, \quad \lambda_2 = 5, \quad \lambda_3 = 6, \quad \lambda_4 = 10;$$

furthermore:

$$M = 8; \quad \delta = \frac{|\varepsilon + \gamma|}{32}$$

thus:

$$|\lambda_k - \mu_m| \leq 48 \left\{ \frac{|\varepsilon + \gamma|}{32} \right\}^{1/4} \qquad 1 \leq k, m \leq 4.$$

Results have been compiled in Table 3.

Table 3:

$\varepsilon = \gamma$	$\lvert\lambda_k - \mu_m\rvert$
0.1	13.50
0.5	20.18
1.0	24.00

The obtained assessment is useless, since eigenvalues of the approximate problem are by one order less than the predicted errors. Table 4 contains effective deviations of correlated eigenvalues, to that, however, also the eigenvalue problem of matrix **C** had to be solved.

Table 4:

$\varepsilon = \gamma$	$\lvert\lambda_1 - \mu_1\rvert$	$\lvert\lambda_2 - \mu_2\rvert$	$\lvert\lambda_3 - \mu_3\rvert$	$\lvert\lambda_0 - \mu_4\rvert$
0.1	0.0012	0.0030	0.0033	0.0009
0.5	0.0300	0.0683	0.0768	0.0215
1.0	0.1218	0.2212	0.2568	0.0862

Results — and analysis of the error assessment formula — showed this method to rather poorly assess effective deviations, imposing to find another idea for an error assessment of merit.

4.2.2 Eigenvalue error assessment by lower and upper limit approximation

The suggested method is safely efficient if all eigenvalues of matrix **C** for the quasi-symmetric case are real and single. The condition of real eigenvalues is met in conservative mechanical systems described by (9): in the actual case, beside this, the second condition prevails: all eigenvectors are single.

For a better understanding of the train of thought, let us expand the characteristic polynomial of matrix **C** in the example (introducing notation):

$$p(\eta) = [(8 - \eta)^2 - 4]\,[(4 - \eta)^2 - 1] - \varepsilon\gamma(8 - \eta)\,(4 - \eta) \tag{11}$$

Representing the independent part of the polynomial exempt from disturbances (ε and γ) as an (actually) fourth-degree polynomial (Fig. 3) and, in the same coordinate system, the part depending on the disturbance as a quadratic function, then varying the disturbance value the part independent of the disturbance will be cut in different places. Projections of intersection points on the η-axis define the (exact) eigenvalues belonging to the actual disturbance.

Do not consider the disturbance in the initial quasi-symmetric problem as basic disturbance, referred to by a subscript 0 (ε_0, γ_0). Remind that $\varepsilon = \gamma$ is not always met in dynamic problems, namely the structure to be modelled may be a non-conservative one, but existence of the equality is not a condition of applying the method.

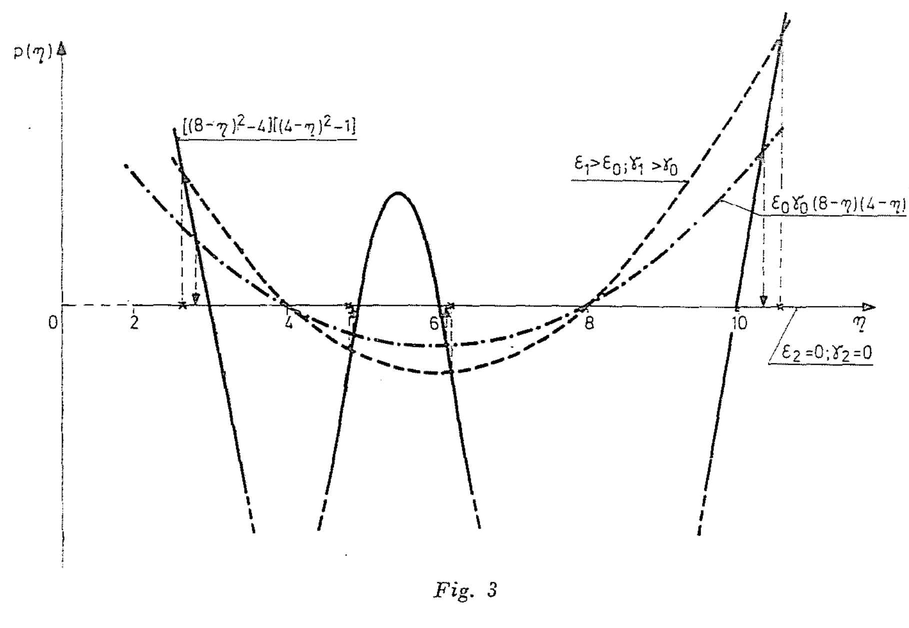

Fig. 3

If in the actual case $\varepsilon_1 \gamma_1 > \varepsilon_0 \gamma_0$ then half of the eigenvalues of the initial problem will be approximated from below. and the other half from above. Similarly. for $\varepsilon_2 \gamma_2 < \varepsilon_0 \gamma_0$ then all eigenvalues will be approximated from the opposite side than in the former case. all in all, the modified disturbance variation ($\varepsilon_1 \gamma_1 \approx \varepsilon_2 \gamma_2 \approx \varepsilon_0 \gamma_0$) according to the unequality $\varepsilon_2 \gamma_2 < \varepsilon_0 \gamma_0 < \varepsilon_1 \gamma_1$ is a possibility to bilaterally approximate the wanted eigenvalues. Obviously, provided $|\varepsilon_i \gamma_i - \varepsilon_0 \gamma_0| \rightarrow 0$ accuracy of interception hence of approximation fast increases, that is simply the reflection of the continuous dependence of solutions on initial data, hence also this assessment procedure is a "sharpening" one.

For ε_0 and γ_0 rather close to zero, zeroing one disturbance leads to a decomposing, thus, ready-to-solve problem. Solution of the so-called "overdisturbed" problem intercepting the eigenvalue, is, however, as difficult as that of the basic problem. Nevertheless, determination of lower-upper approximate eigenvalues may be of importance for the solution of so-called synthesis problems.

A familiar dynamic problem in design is to select free parameters of a mechanical system to meet given specifications (e.g. stability). The solution method is then usually serial analysis, when systematic overall examination of a set of structures is applied to find the closest one — or maybe just the accurate one — meeting the preconditions. In this case also the effect of disturbances of the $|\varepsilon_0 \gamma_0|$ order can be reckoned with between strict lower and upper limits, so that eigenvalues of their approximations from below and from above are examined in a minor part of the set of structures, while in the greatest part, a one-sided (easy) approximation at engineering level will do it.

Summary

In static and dynamic analyses, structural symmetry can be used for reducing the volume of computations. In static analyses, quasi-symmetry can be reduced to a symmetric problem, of reduced laboriousness. but in dynamic analyses it is only approximated. Now, closeness of the approximation can be appreciated by error assessment relying on the disturbance method.

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