# A STOCHASTIC MODEL OF INVENTORY CONTROL STORAGE SYSTEM BEHAVIOUR WITH RESPECT TO QUEUES

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## 1. Introduction

With the help of the stochastic model described in this paper the behaviour of storage systems controlled by inventory strategies (s, S) and (S, S)* can be described and evaluated. Analysis of the behaviour of these systems can be accomplished on two different levels depending on whether the demands or inputs unsatisfied because of the unsatisfactory stock-level and capacity of the storage service systems will be rejected and "lost" or, in the alternative case, the system has a "memory" and rejected demands and inputs accumulate into waiting queues.

Our investigations carried out on the basis of the results given below involve an integration and extension of the possibilities offered by *inventory control theory* and *storage-technological investigations*. They indicate that an inventory control theory which reflects changes in the stock-level of the store only in a highly general way is rather less suitable for the description and accurate understanding of those complex material and information flows that "go on" in a particular storage system with its actual technological conditions. The results obtained by inventory control models are at variance with the actual state of affairs mostly because the capacities or occasional failures of the involved input and output service systems are ignored.**

As the integration of these factors into a model would require a much more complex inventory theory, it seems more suitable to construct system models which are sufficiently general, account for the "internal" processes in the store and, through an extension of storage-technological investigations, are able to take into consideration the influence of particular storage-techno-

* More exactly, the inventory policy called "bringing the level up to level S" (strategy (S, S) hereinafter).

** Of course, this fact does not diminish the enormous significance of inventory control theory in drafting the stock-building policies of a company, in preparing decisions of stock-building and in outlining optimal strategies in the course oflarge-scale stock-building investigations.

logical blueprints on the whole storage system. The complex model (algorithm) presented in this paper is (1) time-dependent, (2) controlled by the stochastic flows of demands, (3) contains "memory" (waiting queues) and (4) feedback (inventory control). Granted sufficient and suitable practical information, it can be used for a realistic description of the "life" of the system with respect to the influence of the inner processes of the store on its behaviour.

The application of the model and the algorithm can have a special significance in planning aspects of control and system behaviour of high-storage systems. The model applies to these aspects since it describes the dynamic behaviour of the system on the basis of a previous estimation of the basic parameters of the store such as the capacity of the store, the capacity of its service systems and the characteristics of its inventory control strategy. The results of the complex model can be applied even in the automated control of storage systems, thus providing possibilities for the permanent development of storage sytems.

## 2. Characteristics of Inventory Control Storage System Behaviour

The notion and functions of storage systems, the basic questions of their behaviour and "conduct", the model constructed to investigate the behaviour of the system and, also, the principles of constructing such a model were discussed in our previous papers [16] [17]. In the following we discuss only those notions and questions which are necessary for the investigation of inventory control storage systems and for the evaluation of their behaviour.

*The behaviour of the inventory control storage system* (*viz.* the canges in its characteristics in time) is determined by the characteristics of the information and material flow in the supplying and consuming systems, by the chosen inventory control strategy and its parameters and by the changes of the store's own abilities (Fig. 1).

*The (external) functioning of the storage system* means that the store takes care of the appropriate satisfaction of emerging demands by a continuous control of stock-level according to a certain stock-building policy. This is achieved by giving an order which is either dependent on stock-level or constant in (generally) determined periods of time depending on the inventory strategy and its parameters.

*The (internal) functioning of the store* can be interpreted as follows. The ordered amount of goods (input flow of materials), that can be delivered "at once" or "protractedly",* during several phases of time, reaches its appointed

* This is roughly synonymous with the term "periodical input"; see [9].

storing place (according to some plan of indoor arrangement) through the input service system. The output flow of materials controlled by the flow of demands leaves the store through the output service system according to a certain output strategy.

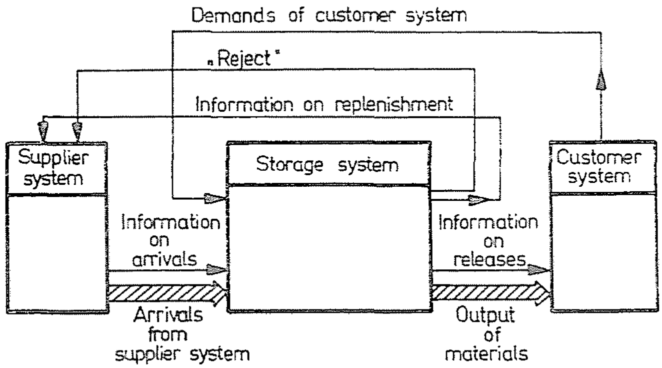


*Fig. 1.* Storage system in its relation to the flow of materials and information

*The input and output flows of materials* are limited by the throughput capacity of the service systems. The throughput capacity depends on the capacity and reliability of the service systems and also on the output and indoor-arrangement strategies applied. Therefore in our model throughput capacity is considered as a random variable. In the model we assume the service system to be divided into two independently functioning systems *viz.* the output and input system. If this is not the case, the available capacity is divided between the input and the output systems,* (with regard to length of waiting queues and the level of stock) by the *service strategy* [17], but the probability distributions of the capacity divided between input and output tasks are assumed to be available from the service strategy.**

*Our model of inventory control storage system behaviour* discussed in this paper takes into consideration the probability distribution of service system capacity (the problems involved in the definition of the probability distribu-

* From the viewpoint of simulating storage system behaviour, the assumption that there is an output system different from the input service system can be either fully justified (e.g. stores with silo type storage system) or partly justified (e.g. specific — in German terminology: Komissionierlagern — stores) or quite frequently it can be considered a good approximation (e.g. throughput stores).

** Generally, this can be simulated by giving a random variable that controls the input and output intensity values in particular phases. Our investigations concerning this method (which are based on the most general strategy according to which service strategy divides the given value of the entire service capacity between the input and output processes in proportion to the input and output queues, on the basis of the actual level of stock in a given phase and the holding capacity of the store respectively) see in [17].

tion of capacities are discussed in [10]) and thus describes the internal "life" of the store, viz. the effect of service system capacity (indirectly, of its internal functioning) on storage system behaviour. After the integration of specific inventory control strategies into the model, there is a possibility to construct "memory-type"* models, as a further specialization of model [16]. In this model the system "remembers" the information concerning demands left unsatisfied because of the unsatisfactory level of stock and capacities, and attempts to satisfy them (the emerging queue of demands) in subsequent phases. (From the viewpoint of queueing the consumer system is a "patient customer".) The model describes the characteristics of emergent input waiting queues that arise because of the unsatisfactory level of storage and capacity of the service system.

## 3. Practical Problems of The Application of Inventory Control Model and The Place of Inventory Control in The Description of Storage Systems

Discussions of inventory control models have brought important results which are widely applied in the organization and planning of various economic processes. Models applying ordering strategies of (s, S) and (S, S) types can be considered the most important and most widely known.

Investigations based on inventory control models and on the stochastic analysis of the dynamic changes of the stock are carried out to determine optimal strategy parameters in case of which the overall cost computed with respect to the respective costs of storage, ordering, and penalty on shortages, etc. is minimal.

The system model to be discussed here uses a different approach and concentrates on problems of a different nature. It attempts to describe storage system behaviour in a more comprehensive way. Our task is to construct a model which describes inventory control storage system behaviour with respect to the capacity of its service system and the effect of input and output queues on the behaviour of the system. In the case of the above mentioned integrated description of storage system behaviour that takes into consideration technology and the flow of demands and of materials as well, the simplifying approach of models (s, S) and (S, S) often cannot be applied.

This is shown by the fact that the amount of repeat order goods (which usually arrive "at once") is usually unlimited. So, in the optimal case even those repeat orders are permitted which would be either impossible to store or would require a service system with an enormous prohibitive capacity.

* Two types of storage system models can be investigated according to their type of "memory". Models with restricted or unrestricted memory can be spoken of. In case of the former, "restricted" means that the store is no longer able to remember the input quantities of materials and the queueing demands beyond a certain level Q (these demands "get lost"). Obviously, the latter type has no restriction of this sort.

*According to inventory control model approaches*, this means that storage costs which obviously include investment and other continuous costs of the service system are not constant, (as these models assume them to be), but depend to a great extent on the parameter values of inventory control strategies.

These suggest that, in accordance with the approach and modelling principles [16], inventory control strategies (in our case, strategies of the (s, S), and (S, S) type) should be integrated into the general model of storage system behaviour. Thus, our primary aim is to integrate the "black box" approach of inventory control and "dam" theory and, on the other hand, investigations of storage-technology into a unified whole and, by a consideration of input and output waiting queues and the particular inventory control strategy involved, to describe storage system behaviour and make it suitable for evaluation.

## 4. A Stochastic Model of Inventory Control Storage System Behaviour

### 4.1. *Conditions of Simulating "Memory-type" Inventory Control Storage System Behaviour*

In the model, the characteristics of the behaviour of a "memory-type" inventory control storage system which functions on the basis of strategies (s, S) and (S, S) are the following:

1. The behaviour of the system in relation to time is investigated in discrete phases.

2. The system receives deliveries from a supplier system at a frequency and in a quantity determined by the inventory control strategy involved.

3. Orders are given by the inventory control strategy on the basis of the *"abstract"* level of stock at* the beginning of each phase when the level of stock reaches level $s$ (in the case of strategy (s, S)), or in every $T (T \geq 1)$ phase (in the case of strategy (S, S)). In case of both strategies the system orders an amount of materials which makes up for the difference between level S and the "abstract" level of stock and this amount of materials is delivered by the supplier system by the end of each phase. (The case of so called "protracted" deliveries which is a better approximation of actual processes can also be incorporated into the model.) [9]

* By "abstract" level of stock a modified version of the notion "level of stock" is meant as it is used in inventory control theory. Here it differs from the notion of actual stock in [16] and in this paper so far. In inventory control theory the level of stock (K) is jointly determined by the level of stock "on the books" ($k$), the amount of actual (physical) stock ($f$) and the amounts of remaining unmet demands ($m$). That is, $K = f - k + m$.

In our case the abstract level of stock is jointly determined by the emerging and queueing demands or inputs in the phase concerned and the actual level of stock. The difference between these two approaches is not very significant if the length of queues is significantly smaller than numbers $s$, $S$ in the strategy (which is a requirement for "good" stores).

4. In each phase the system receives demands from a consumer system. Absence of demands is considered as a 0 amount of demands.

5. The storage system is assumed first to receive the materials arriving from the supplier system in the given phase and the input waiting queue that remained from the previous phase "to the best of its abilities", that is, to the extent it is made possible by the level of stock, the capacity (and, also, the capacity of the input service system). Then, by the end of the phase, the system satisfies the total amount of emerging demands during the phase, that is, the demands that arrive in the phase and the waiting queue of demands unmet in the previous phase "to the best of its abilities" (that is, to the extent the level of stock and capacity make it possible). (This condition can be met by a suitable choice of phases.)

6. The storage system does not release more materials than required by the demands and the amount of materials in excess of its capacity, that is, the amount that cannot be received because of the level of stock of the moment and capacity, is "compelled to wait".

7. In our model of storage system behaviour, level of stock and the output amount of materials in phase $t$ are considered independent of waiting demands and/or emerging inputs in phases $t-2$, $t-3$, . . .etc., that is, earlier states of the system can influence later states only through the present state and the previous one.*

8. For a demand in excess of the level of stock in a given phase the store releases its whole stock (a modelling condition widely used in literature).

### 4.2. *The Definition and Parameters of Input and Output Queues in a Storage Systems*

Paper [16] discussed in detail the general (environmental, technological) parameters of storage system behaviour.

All the system parameters together with parameters characterizing queues given below are summarized in Table 1. From now on only the characteristics of waiting demands and inputs will be concerned with. Let $\omega(t)$ denote the amount of waiting demands (the output queue of demands) in phase $t$ in arbitrary units of materials.

From the above it is obvious that $\omega(t)$ is a discrete random variable if $t$ is given. Its probability distribution will be denoted by

$$h_a(t) = P(\omega(t) = a). \qquad a = 0, 1, 2, \ldots$$

* In the case of Markov chains, earlier states of the system can influence later states only through the present state. [13] (For example model [16] for the "reject-type" storage system behaviour). In case of queues earlier states of the system can influence later states only through the present phase and the previous one.

Accordingly, the "reject-type" model of storage system behaviour (with controlled homogeneous and inhomogeneous Markov chains and stochastic automata) has no "memory", while in further cases stochastic models "with memory" can be spoken of.

**Table 1**

Storage system parameters and characteristics

| Nature of system parameters | Assumed system parameters | | | | | | | Computed system parameters | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Classes of system parameters | Environmental parameters | | Technological parameters | | | Strategy parameters | | Behaviour parameters | | | | | |
| Stochastic process | Demand of customer system (expressed in arbitrary units) in phase t | Quantitiy of arrivals, ordered in previous phase (expressed in arbitrary units) in phase t | Input service capacity of store (expressed in arbitrary units) in phase t | Output service capacity of store (expressed in arbitrary units) in phase t | (Containing) capacity of store in arbitrary units | Parameters of inventory control strategy according to the type of strategy chosen: S: stock level to which bring up the level of stock (expressed in units) if it falls below s: ordering level in arbitrary units T: ordering period (expressed in number of phases) | Amount of initial stock, i. e. the quantity of materials in store (expressed in arbitrary units) in phase O | Amount of stock in store (expressed in arbitrary units) in phase t | Output of store (expressed in arbitrary units) in phase t | Length of output queue (amount of waiting demands) in phase t | Amount of total demand (sum total of demands averaging in present phase and waiting demands from previous phase) in phase t | Length of input queue (number of units waiting for input) in phase t | Total amount of input (number of units waiting from previous phase and number of units arriving in present phase) in phase t |
| Notation | $\xi$ (t) discrete random variable | $\eta$ (t) discrete random variable | $\lambda_{in}$ (t) discrete random variable | $\lambda_{out}$ (t) discrete random variable | C positive integer | s, S, T positive integer | $\alpha$ (O) discrete random variable | $\alpha$ (t) discrete random variable | $\beta$ (t) discrete random variable | $\omega$ (t) discrete random variable | $\xi^*$ (t) discrete random variable | $\varkappa$ (t) discrete random variable | $\eta^*$ (t) discrete random variable |
| Distribution | $p_l(t) = P[\xi(t) = l]$ $l = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $r_k(t) = P[\eta(t) = k]$ $k = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $u_n(t) = P[\lambda_{be}(t) = n]$ $n = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $v_z(t) = P[\lambda_{ki}(t) = z]$ $z = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | — | — | $q_h(O) = P[\alpha(O) = h]$ $h = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $q_h(t) = P[\alpha(t) = h]$ $h = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $w_s(t) = P[\beta(t) = s]$ $s = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $h_a(t) = P[\omega(t) = a]$ $a = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $p_l^*(t) = P[\xi^*(t) = l]$ $l = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $g_b(t) = P[\varkappa(t) = b]$ $b = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ | $r_k^*(t) = P[\eta^*(t) = k]$ $k = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$ |
| Probability distribution vector | **p** (t) | **r** (t) | **u** (t) | **v** (t) | — | — | **q**(O) | **q** (t) | **w** (t) | **h** (t) | **p*** (t) | **g** (t) | **r*** (t) |

Let discrete random variable $\xi^*(t)$ denote the total amount of demands in phase $t$ (that is, the sum total of the amount of waiting demands in phase $t-1$ and the demands emerging in phase $t$) in units of materials. Let $p_l^*(t)$ denote the distribution of total amount of demands $\xi^*(t)$ as a discrete random variable:

$$p_l^*(t) = P(\xi^*(t) = l) \qquad l = 0, 1, 2, \ldots$$

Let $\varkappa(t)$ denote the amount of materials waiting for reception (the length of the input queue), in phase $t$, in units of materials. If $t$ is given, $\varkappa(t)$ is a discrete random variable and its probability distribution is:

$$g_b(t) = P(\varkappa(t) = b) \qquad b = 0, 1, 2, \ldots$$

Let discrete random variable $\eta^*(t)$ denote the total amount of input materials arriving in phase $t$ (that is, the sum total of the amount of materials unreceived in phase $t-1$ and the input of materials in phase $t$) in units. Let

$$r_k^*(t) = P(\eta^*(t) = k) \qquad k = 0, 1, 2, \ldots$$

the probability distribution of $\eta^*(t)$.

The system parameters summarized in Table 1 are also given in the model diagram of Fig. 2 that illustrates functioning of the system.

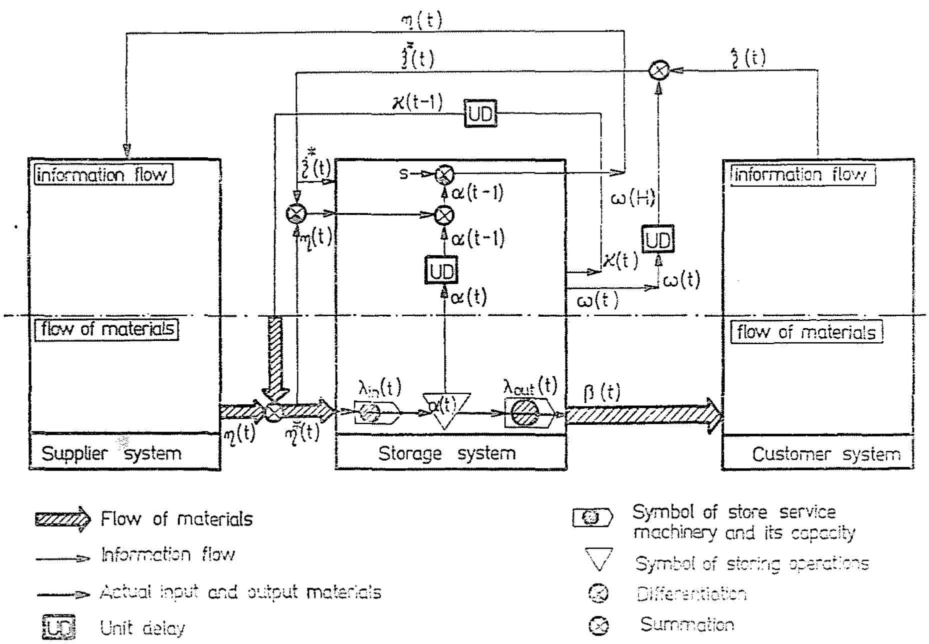


*Fig. 2.* Parameters and model for storage system behaviour (of (S, S) type)

### 4.3. *A Formal Model for The Behaviour of Storage Systems with Strategies (s, S) and (S, S) (in Transient State)*

#### 4.3.1. *Transition and Output Probability Matrices. Probability Distributions of Input and Output Queues.*

System behaviour will be described and evaluated on the basis of the parameters of inventory control storage systems given and characterized in Table 1.

To determine the evaluating parameters which characterize storage system behaviour first we have to calculate the conditonal probability distributions

$$P[h(t+1)/h(t),\ l^*(t)] \quad \text{and} \tag{1}$$

$$P[s(t)/h(t),\quad l^*(t)] \tag{2}$$

and probability distribution

$$P(\omega(t)=a) \quad \text{and} \tag{3}$$

$$P(\varkappa(t)=b) \tag{4}$$

which characterize storage system behaviour.

Here (1) denotes the conditional probability that level of stock will be $h(t+1)$ in phase $(t+1)$ if the level of stock of the store in phase $t$ is $h(t)$ and the total amount of demands is $l^*(t)$, (2) denotes the probability that $s(t)$ will be the output amount of materials in phase $t$ if the level of stock is $h(t)$ in phase $t$ and the total amount of demands is $l^*(t)$. (3) denotes the probability of the amount of waiting demands (the length of output queues) according to 4.2., while (4) gives the probability distribution of the amount of input materials waiting for reception (the length of input queues) in phase $t$.

First we have to determine two matrices consisting of conditional probabilities for the explicite calculation of values (1) and (2).

*Theorem 1**

The storage system is assumed to satisfy conditions 1—8 given in 4.2. In this case the level of stock of the store depending on the total amount of demands determines a transition probability matrix system. Choosing an arbitrary stochastic matrix $\mathbf{M}_l(t)$ from this system, the matrix-elements will be determined only by the distribution of random variables $\xi(t)$, $\eta(t)$, $\lambda_{in}(t)$ and $\lambda_{out}(t)$ the capacity $C$ of the store, and the distribution of random variables

* Proof is not given in detail since it coincides with the proof of a similar theorem in [16].

$\omega(t-1)\ \varkappa(t-1)$ of the previous phase, and they will correspond to the probabilities described in (1). Element $\mathbf{M}_l(t)$ of stochastic matrix $m_{ij}\ (l)$ can be determined as follows*

where

$$m_{ij}(l) = \sum_{z=0}^{l-1} v_z d(i,j) \mid z) + d(i,j \mid l) \sum_{z=l}^{C} v_z$$

$$d(i,j \mid z) = \begin{cases} \sum_{\nu=0}^{z-i} (u_\nu + r_\nu^*) - \left(\sum_{\nu=0}^{z-i} r_\nu^*\right)\left(\sum_{\nu=0}^{z-i} u_\nu\right) \text{ if } j=0, 0 \le i \le C, z-i \ge 0 \\ \sum_{j=C}^{\infty}\left[u_{j-i+z} + r_{j-i+z}^* - r_{j-i+z}^* u_{j-i+z} - \sum_{\nu=0}^{j-i+z-1} u_{j-i+z} r_\nu^* + r_{j-i+z}^* u_\nu)\right] \\ \qquad \text{if } j=C, 0\le i \le C, C-i+z>0 \\ u_{j-i+z} + r_{j-i+z}^* - r_{j-i+z}^* u_{j-i+z} - \sum_{\nu=0}^{j-i+z-1} (u_{j-i+z} r_\nu^* + r_{j-i+z}^* u_\nu) \\ 0 \text{ otherwise} \qquad \text{if } 0<j<C, 0 \le i \le C, j-i+z \ge 0 \end{cases} \quad (5)$$

where

$$r_k^*(t) = \sum_{b=0}^{\infty} r_{k-b}(t)\, g_b(t-1) \quad (6)$$

is the probability distribution characterizing the input amount of materials in phase $t$ that is the convolution of the probability distributions of both the input queue of phase $t-1$ and the input amount of materials in phase $t$. The probability distribution characterizing the input amount of materials $r_k(t)$, depending on the strategies used, can be determined according to the following theorem.

*Theorem 2*

If conditions 1 to 8 of 4.1. are satisfied, the probability distributions characterizing the amount of materials ordered by the storage system at the end of phase $t-1$, (that is, the input amount of materials in phase $t$) can be determined by the following relationships: In case of strategy (s, S)

$$r_k(t) = \begin{cases} \sum_{i=0}^{\infty} q_i(t-2) \sum_{l=0}^{\infty} p_l^*(t-1)\, r_{l-i+S-k}^*(t-1) \quad \text{if} \quad S-s \le k \le S+C \\ \sum_{h=s+1}^{C} \sum_{i=0}^{\infty} q_i(t-2) \sum_{l=0}^{\infty} p_l^*(t-1)\, r_{l-i+h}^*(t-1) \quad \text{if} \quad k=0, s<h\le C \\ 0 \quad \text{otherwise} \end{cases} \quad (7)$$

* In the above and following relationships the convention is applied that if a summation or index becomes negative the formula is to be zeroed; for the sake of simplicity, the (phase) variable "$t$" has been omitted. For convenience's sake the maximum length of Queues is limited to C.

or in case of strategy (S, S)

$$r_k(t) = \begin{cases} \sum_{i=0}^{\infty} q_i(t-2) \sum_{l=0}^{\infty} p_l^*(t-1)\, r_{l-i+S-k}^*(t-1_S) & \text{if} \quad 0 \leq k \leq S+C \\ 0 \quad \text{otherwise} \end{cases} \tag{8}$$

*Proof*

From evident considerations based on the theory of probability and the definition of strategies (s, S) and (S, S) it is obvious that if the "abstract" stock-level of the store (which is relevant from the point of view of repeat orders) in phase $t-1$ is determined by probability distribution vector $\mathbf{q}^*(t-1)$, then the probability distribution characterizing the amount of arrivals in the next phase (the amount of materials reordered in phase $t-1$) can be calculated with the help of the following relationships:

$$r_k(t) = \begin{cases} q_{S-k}^*(t-1) \text{ if } S-s \leq k \leq S+C \\ \sum_{k=s+1}^{C} q_h^*(t-1) \quad \text{if } k=0 \\ 0 \quad \text{otherwise} \end{cases} \qquad r_k(t) = \begin{cases} q_{S-k}^*(t-1) \text{ if } 0 \leq k \leq S \\ 0 \qquad \text{if} \quad k = S \end{cases} \tag{9}$$

in case of strategy (s, S) in case of strategy (S, S)

The abstract level of stock $\alpha^*(t)$ can be calculated in our case by the addition of the actual level of stock, the waiting and emerging demands and the arrivals. Thus, in this case, the probability distribution $\alpha^*(t)$ of $q_h^*(t)$ can be determined by the convolution of probability distributions $q_h(t-1)$ $r^*(t)$ and $p_l^*(t)$ with the help of the formula:

$$q_h^*(t) = \begin{cases} \sum_{i=0}^{\infty} q_i(t-1) \sum_{l=0}^{\infty} p_l^*(t)\, r_{l-i+h}^*(t) & \text{if} \quad -C \leq h \leq 2C, l-i+h \geq 0 \\ 0 \quad \text{otherwise} \end{cases} \tag{10}$$

The theorem follows from relationships (6), (7), (8)* If the investigation of

* Drafting of the inventory strategy could be even more realistic if the determination of the ordered amount of materials involved a joint consideration of the abstract level of stock in the previous phase and the capacity of service systems. In addition to these the so-called "periodical" inputs could also be taken into consideration. A consideration of the statistical characteristics of the capacity of the supplying and input systems that modify the flow of materials, etc. would provide an additional means for expanding the model.

the level of stock from the viewpoints of repeat orders is not accomplished in each phase, in these phases probabilities $r_k(t)$ will simplify according to the relationship

$$r_k(t) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise}, \end{cases}$$

*Theorem 3**

The storage system is assumed to satisfy the conditions in 4.2. In this case the amounts of material released in each phase can be characterized by a matrix $N_l(t)$, the element $n_{ij}(l)\,[t]$ of which is equal to the probability in (2). The stochastic elements of matrix $N_l(t)$ (which also depend on the parameters given in Theorem 1) are determined by the formulas:

$$n_{ij}(l) = \sum_{z=0}^{\infty} v_z d(i, j \mid l, z) \tag{11}$$

where

$$d(i, j \mid l, z) = \begin{cases} u_{j-i} r^*_{-i} + \sum\limits_{k=j-i+1}^{\infty} (u_{j-i} r^*_k + r^*_{j-i} u_k) & \text{if } j-i \geq 0 \text{ and } j < l \text{ and } l \leq z \text{ or} \\ & j - i \geq 0 \text{ and } j < z \text{ and } z < l \leq C \\ \left(\sum\limits_{k=j-i}^{\infty} r^*_k\right) \left(\sum\limits_{k=j-i}^{\infty} u_k\right) & \text{if } j - i \geq 0, j = l, l \leq z \leq C \\ 1 & \text{if } j - i \leq 0, j = l \leq z \leq C \text{ or} \\ & j - i \leq 0, j = z < l \\ \sum\limits_{v=z}^{l-1} \left[u_{v-i} r^*_{v-i} + \sum\limits_{k=v-i+1}^{\infty} (u_{v-i} r^*_k + r^*_{v-i} u_k)\right] + \left(\sum\limits_{k=l-i}^{\infty} r^*_k\right) \left(\sum\limits_{k=l-i}^{\infty} u_k\right) & \\ 0 \quad \text{otherwise} & \text{if } j - i \geq 0 \text{ and } j = z < l \end{cases}$$

Here probabilities $r^*_k(t)$ can also be determined on the basis of formulas (6), (7) and (8).

Probability distribution $g_b(t)$ given among the system parameters and already applied (implicite) in the previous theorem (whose elements are probabilities characterizing the length of input queue $\varkappa(t)$ in phase $t$), can be determined by the following theorem.

* Proof is not given in detail since it coincides with the proof of a similar theorem in [16]

*Theorem 4*

An element $g_b$ (3) of the probability distribution concerning the input amount of materials waiting for reception in phase $t$ can be calculated if the following recursive relationship is applied:

$$g_b(t) = \begin{cases} \sum_{h=0}^{C} q_h(t-1)\left[\sum_{v=0}^{C-h-1} u_v(t) \sum_{k=0}^{n} r_k^*(t) + \sum_{k=0}^{C-h} r_k^*(t) \sum_{v=C-h}^{\infty} u_v\right] & \text{if } b=0 \\ \sum_{h=0}^{C} q_h(t-1)\left[\sum_{n=0}^{C-h-1} u_n(t)\, r_{n+b}^*(t) + r_{C-h+b}^*(t) \sum_{n=C-h}^{\infty} u_n(t)\right] & \text{if } b>0 \\ 0 \quad \text{otherwise} \end{cases} \tag{12}$$

*Proof*

Let $P(b/h, n)$ be the conditional probability that the waiting amount of materials not received is $b$ in phase $t$, if the level of stock at the "beginning" of phase $t-1$ is $h$ and the input capacity of the storage system is $n$ (that is, either the amount of materials exceeding level $z$ is "compelled to wait", or the amount of the input materials and the level of stock exceed the capacity of the store). Accordingly, conditional probabilities $P\,(b/h, n)$ can be determined as follows.

$$P\big(b(t)\,|\,n(t), h(t-1)\big) = \begin{cases} \sum_{k=0}^{n} r_k^*(t) & \text{if } b=0 \text{ and } C-h>n \\ r_{n+b}^*(t) & \text{if } b>0 \text{ and } C-h>n \\ \sum_{k=0}^{C-h} r_k^*(t) & \text{if } b=0 \text{ and } C-h\le n \\ r_{C-h+b}^*(t) & \text{if } b>0 \text{ and } C-h\le n \\ 0 \quad \text{otherwise} \end{cases} \tag{13}$$

Applying the formula of total probability twice for distribution $g_b(t)$ yields

$$g_b(t) = \sum_{n=0}^{\infty} u_n(t) \sum_{h=0}^{\infty} q_h(t-1)\, P\big[b(t)\,|\,n(t), h(t-1)\big] \tag{14}$$

from which, after having rearranged it, we get the probability distribution in the theorem for the length of the input queue.

The statistical characteristics of the output queues of the system (that is, the amount of waiting demands) can be defined according to the following theorem.

*Theorem 5*

An element $h_a$ of the probability distribution of the amount $\omega(t)$ of waiting demands (which corresponds to relationship (4)) can be calculated in phase $t$ with the help of the recursive formulas:

$$h_a(t) = \sum_{l=a}^{\infty} w_{l-a}(l)[t]\, p_l^*(t) \tag{15}$$

where

$$w_{l-a}(l)[t] = \sum_{h=0}^{\infty} q_h(t-1)\, n_{l-a,h}(l)[t] \tag{16}$$

and

and $p_l^*(t) = \sum_{a=l}^{\infty} p_{l-a}(t) \cdot h_a(t-1)$, where $n_{ij}(l)\,[t]$ element of matrix $N_l(t)$ (17)

*Proof*

It is obvious that conditional probability $w_{l-a}\,(l)\;[t]$ is the probability that in phase $t$ the system will release an amount $(l-a)$ of materials if the total amount of demands is $l$. This is the same as the conditional probability that the amount of demands unmet in phase $t$ is $a$, if the total amount of demands (demands emerging in this phase and the waiting demands from the previous phase taken together) is $l$ in phase $t$.

The latter conditional probability distribution can be calculated with the help of relationship (16) where stochastic matrix $N_l(t)$ consisting of conditional probabilities $n_{ij}(l)\;[t]$ can be defined on the basis of relationship (11). The relationship to be proved can be obtained (on the basis of the probabilities of the total amount of demands in the given phase) through the application of the formula of total probability.

### 4.3.2. *Transient-state System Behaviour and Possibility of System Identification*

The results obtained so far permit to state the calculations for determining all the probability distributions of the system and, also, the evaluational parameters derived from these distributions for any arbitrary phase.

1. The probability distribution vector $\mathbf{r}^*\,(t)$ in phase $t$ can be obtained by the convolution of the probability distributions of input queue $g_b(t-1)$ of the previous phase

$$\mathbf{r}^*(t) = \mathbf{r}(t) * \mathbf{g}(t-1)$$

The probabilities $r_k\,(t)$ belonging to the various inventory policies can be obtained from theorem 2.

2. In each phase probability distribution vectors $\mathbf{u}(t)$, $\mathbf{v}(t)$, $\mathbf{h}(t)$ are considered as given. Probability distribution vector of total amount of de-

mands $\mathbf{p}^*(t)$ is the convolution of probability distributions of waiting demands $h_a(t-1)$ from the previous phase and the demands $p_l(t)$ in the present phase.

$$\mathbf{p}^*(t) = \mathbf{p}(t) * \mathbf{h}(t-1)$$

of theorems 1 and 3.

4. According to theorem 5, probability distribution $\mathbf{N}_l(t)$ of waiting demands that remain by the end of the present phase can be defined on the basis of stochastic matrix $\mathbf{q}$ $(t-1)$, the probability distribution of level of stock $h_a(t)$ in the previous phase and probability distribution of the total amount of demands in the present phase.

5. The probability distribution vector $\mathbf{w}$ $(t)$ at the end of the present phase and the probability distribution vector $\mathbf{q}$ $(t)$ of the input amount of materials in the phase can be derived from the definitions of stochastic matrices $\mathbf{M}_l(t)$ and $\mathbf{N}_l(t)$ as the theorem of total probability.*

$$\mathbf{q}^T(t) = \mathbf{q}^T(t-1)\,\mathbf{M}(t) \tag{18}$$

where

$$\mathbf{M}(t) = \sum_{l=0}^{\infty} p_l^*(t)\,\mathbf{M}_l(t) \tag{19}$$

$$\mathbf{w}^T(t) = \mathbf{q}^T(t-1)\,\mathbf{N}(t) \tag{20}$$

where

$$\mathbf{N}(t) = \sum_{l=0}^{\infty} p_l^*(t)\,\mathbf{N}_l(t). \tag{21}$$

6. The probability distribution vector of the input waiting queue at the end of the given phase can be calculated by theorem 4. The process described in 1 to 6 is shown in the form of a flow chart block diagram in Fig 3.

Thus the distribution of random variables $\alpha(t)$, $\beta(t)$, $\omega(t)$ and $\varkappa(t)$ that ultimately determines storage system behaviour can be clearly defined by the theorems and, for each phase, by the procedure described in 1 to 6. That is, an algorithm can be constructed to generate the values characterizing storage system behaviour, if the distribution of initial stock is known and the length of the queues is assumed to be zero in phase 0 (Fig. 4).

*The storage system behaviour model* can be made significantly clearer by describing it with the help of *operator formalism*. (See the operator-connection diagram in Fig. 5.) Operator formalism makes possible new kinds of investigations as well. It gives an opportunity to investigate storage systems even if the amount of a priori information is little. In such a case the store as a complex object has to be *identified*. [11]

* $(T)$ denotes the transposed vector (row vector) as defined in matrix algebra.

The operators in the enclosed part of Fig. 5 determine a "system of operators". It maps the probability distributions of the demands (that is, the "input signal") into the "output signals", that is, the probability distributions of consumer demands satisfied. Thus, if $A_t$ denotes the "system of operators" in Fig. 5, the problem described above can be stated with the help of the relationship $W(t) = A_t[P(t)]$ from the consumer point of view.

| | | |
|---|---|---|
| 1. | $\mathbf{r}(t) = \mathbf{r}[\mathbf{q}(t-2), \mathbf{r}^*(t-1), \mathbf{p}^*(t-1)]$<br>$\mathbf{r}^*(t) = \mathbf{g}(t-1) \ast \mathbf{r}^*(t)$ | Theorem 2. |
| 2. | $\mathbf{p}^T(t) = \mathbf{p}(t) \ast \mathbf{h}(t-1)$ | |
| 3. | $\mathbf{N}_l(t) = \mathbf{N}_l[\mathbf{u}(t), \mathbf{v}(t), \mathbf{r}^*(t)]$ | Theorem 3. |
| 4. | $\mathbf{w}_l^T(t) = \mathbf{q}^T(t-1)\, \mathbf{N}_l(t)$ | |
| 5. | $\mathbf{h}(t) = \mathbf{h}[\mathbf{w}_l(t), \mathbf{p}^*(t)]$ | Theorem 5. |
| 6. | $\mathbf{M}_l(t) = \mathbf{M}_l[\mathbf{u}(t), \mathbf{v}(t), \mathbf{r}^*(t)]$ | Theorem 1. |
| 7. | $\mathbf{M}(t) = \sum_l p_l^*(t)\, \mathbf{M}_l(t)$<br>$\mathbf{q}^T(t) = \mathbf{q}^T(t-1)\, \mathbf{M}(t)$ | |
| 8. | $\mathbf{N}(t) = \sum_l p_l^*(t)\, \mathbf{N}_l(t)$<br>$\mathbf{w}^T(t) = \mathbf{q}^T(t-1)\, \mathbf{N}(t)$ | |
| 9. | $\mathbf{g}(t) = \mathbf{g}[\mathbf{r}^*(t), \mathbf{u}(t)\, \mathbf{q}(t-1)]$ | Theorem 4. |

*Fig. 3.* Algorithm in phase t for calculating storage system behaviour

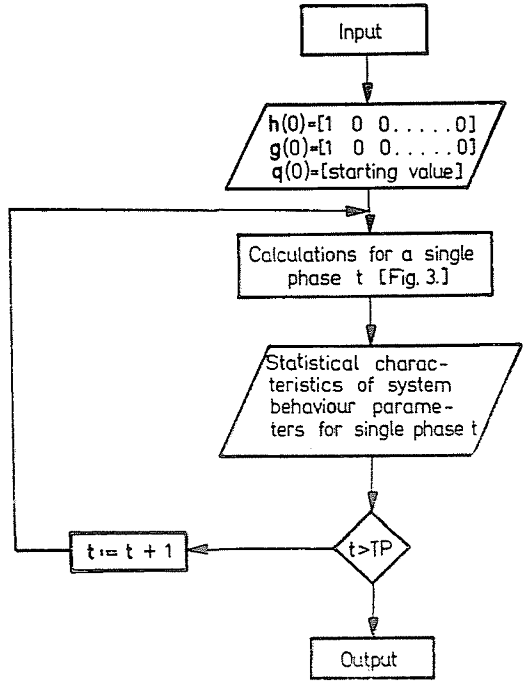


*Fig. 4.* **Process diagram for algorithm of system behaviour**

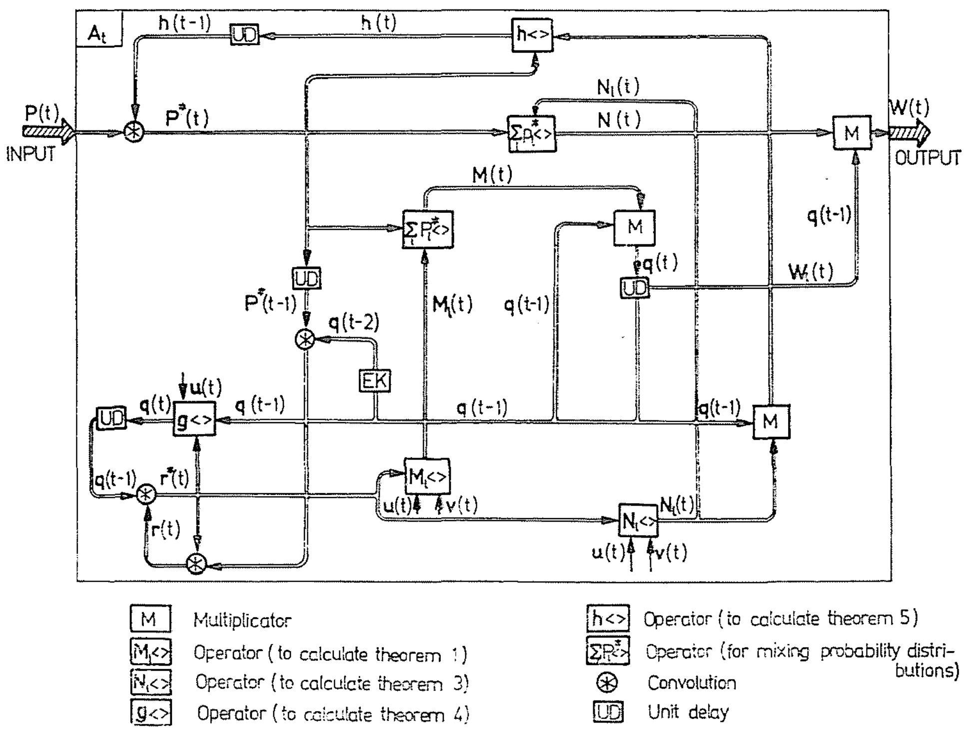


*Fig. 5.* **Operator-connection diagram of stochastic system behaviour model**

**Table 2**

Basic parameters for evaluating storage system behaviour

| Stochastic process | Notation | Statistical characteristics | Notation | Definition |
|---|---|---|---|---|
| Inventory function (stock level) | $\alpha(t)$ | expected value | $E[\alpha(t)]$ | $\sum_{h=0}^{\infty} h \cdot q_h(t)$ |
| | | variance | $D[\alpha(t)]$ | $\sum_{h=0}^{\infty} \{h - E[\alpha(t)]\}^2 q_h(t)$ |
| Waiting demands (output queue) | $\omega(t)$ | expected value | $E[\omega(t)]$ | $\sum_{a=0}^{\infty} a \cdot h_a(t)$ |
| | | variance | $D[\omega(t)]$ | $\sum_{a=0} \{a - E[\omega(t)]\}^2 h_a(t)$ |
| Waiting inputs (input queue) | $\varkappa(t)$ | expected value | $E[\varkappa(t)]$ | $\sum_{b=0}^{\infty} b \cdot g_b(t)$ |
| | | variance | $D[\varkappa(t)]$ | $\sum_{b=0}^{\infty} \{b - E[\varkappa(t)]\}^2 g_b(t)$ |

So far the information needed to formulate "system of operators" $A_t$ was assumed to be a priori given. In case of a system already existing, the system parameters are not known, so they have to be estimated for each phase on the basis of observations and measurements. They will, then, define operator $A_t^*$ which estimates operator $A_t$ in the given phase. Now, in the framework of the theory of identification our problem can be stated as $\mathbf{W}^*(t) = A_t^*[P(t)]$. [11]

For the identification of storage systems further investigations are necessary; Table 3 shows, as an example of the favorable possibility of application, a cross-dispersion function which indicates the closeness of the connection between the series of demands and those of satisfaction and can well be applied in the case of identifying non-linear objects. [11], [12]

### 4.4. *Permanent State of The Behaviour of Storage Systems with Periodic or Time-Independent Demands. Evaluation Problems Involved*

#### 4.4.1 *System Behaviour in Permanent State*

Most storage systems in practice have one thing in common, namely, that in the short run the probability distribution of their capacities $\mathbf{u}(t)$, $\mathbf{v}(t)$ can be considered as "independent of time", or the distribution of demands

**Table 3**

Indices and statistical characteristics for evaluating storage system behaviour

| Name | Notation | Relationship |
|---|---|---|
| Functional reliability of demand satisfaction | $FR_{out}(t)$ | $\frac{E[\beta(t)]}{E[\xi(t)]}$; $\frac{E[\beta(t)]}{E[\xi^*(t)]}$ |
| Functional reliability of input processes | $FR_{in}(t)$ | $\frac{E[\xi(t)] - E[\varkappa(t)]}{E[\eta^*(t)]}$; $\frac{E[\eta^*(t)] - E[\varkappa(t)}{E[\eta(t)]}$ |
| Index of store capacity | $R_k(t)$ | $\frac{E[\alpha(t)]}{C}$ |
| Expected value of storage "overflow" (because of insufficient store capacity) | $E[R(t)]$ | $\sum_{h=0}^{C}\sum_{k=1}^{\infty}\{q_h(t)\, r_{C-h+k}(t)\,[C-h+k]\}$ |
| Probability of "overflow" | $P_t(C)$ | $q_C(t)\ [1-r_0^*(t)]$ |
| Index of "overflow" | m (t) | $\frac{E[R(t)]}{C}$ |
| Index of stock-level oscillation | $i(t)$ | $\frac{D[\alpha(t)]}{E[\alpha(t)]}$ |
| Entropy of inventory function | $H[\alpha(t)]$ | $\sum_{h=0}^{C} q_h(t)\cdot\log q_h(t)$ |
| Conditional entropy of "closeness" of demands and satisfactions | $H[\beta(t)\xi(t)]$ | $\sum_{l=0}^{C}\sum_{s=0}^{C} P_l(t) w_s(l)[t]\log w_s(l)[t]$ |
| Normalized (cross-) dispersion function showing the "closeness" of demands and satisfactions in time (the "rate of stochasticity" of the system) + [11] [12] | $\Theta_{sl}(t+\tau,t)$ | $\left[\frac{\sum_{l=0}^{C} p_l(t)\left[\sum_{s=0}^{C} s\cdot w_s(l_t)[t+\tau] - \sum_{s=0}^{C} s\cdot w_s(t+\tau)\right]^2}{\sum_{s=0}^{C} w_s[t+\tau]\left[s-\sum_{s=0}^{C} s\cdot w_s[t+\tau]\right]^2}\right]^{-1}$ |

in time is of periodic character. Thus a cycle consisting of phase "$k$" can be defined, where the distributions described above are the same in every phase k [16]. (This way seasonal factors can also be included.)

The character of the stochastic model (that is, the fact that the internal and external functioning of the store can be described with an unambiguous algorithm) and, also, certain heuristic considerations make it clear that if the environmental effects described above are periodic, the distributions characte-

* Conditional probability distribution vector $w_s(l_t)[t+\tau]$ in the relationship from: $w_s(l_t)[t+\tau] = \mathbf{q}(t-1)\ \mathbf{M}_l(t)\ \mathbf{M}(t+1)\ \ldots\ \mathbf{M}(t+\tau-1)\ \mathbf{N}(t+\tau)$. Its elements give the probability that the amount of output materials will be s in phase $(t+\tau)$, provided demand in phase t was $l$.

rizing the store will absorb the periodicity of the environmental effects in the long run. Thus, the distribution vectors of each phase $k$ will hardly differ from each other. In the language of mathematics, if each element $k$ of each series of distribution vectors is considered, *this series has a limit value.* Several computer experiments have been made by simulating the periodic behaviour of the environment and consequently that of the system [16]. These experiments have shown that the state of equilibrium (the permanent state of system behaviour) will set in after a relatively short time; the initial (not permanent) behaviour of the system can only be the consequence of the "noise" caused by the fact that the distribution of the initial stock was other than the limit distribution. (See Example).

Investigation of these special systems seems to be significant because, for the evaluation of storage systems, in this case it is obviously sufficient to investigate only the stochastic characteristics (indices) deduced from the limit distributions. Independent and essential parameters are: *the level of stock, the amount of waiting demands* (length of output queue) *and the amount of waiting inputs* (length of input queue) (See Table 2).

During the processes of planning and investigation of storage systems the basic characteristics such as the capacity of the store, the probability distribution of the input and output service systems, the parameters of the inventory strategy, etc. can be evaluated on the basis of a certain choice of these parameters and, if necessary, optimalizing procedures can also be carried out.*

### 4.4.2 *Determination of Expected Value of Ordering Period for System Behaviour with Inventory Policy (s, S)*

If the transition probability matrix and other statistical parameters in a permanent state are known, other important parameters of system behaviour can be calculated beside the given characteristics [5] [7]. For example, in case of a storage system with strategy (s, S) the *average ordering period* (which may extend over more than one phase) can be determined. This can also be considered an essential characteristic of the system. For convenience's sake, a specific case will be presented where unmet input demands in a given phase are rejected.

In this case the expected value of the ordering period can be calculated as follows, assuming to have a periodic or time-independent flow of demands

* Besides the identificational investigations, incorporation of learning algorithms (Tsypkin [15]) would be the most suitable means for the further development of the model. Thus the learning processes of storage system behaviour could be investigated if the probability distribution of the flow of demands is unknown and the essential system parameters would be continuously corrected. (There is also a possibility of incorporating other methods) e.g. the Bellman-Murphy adaptive algorithm operating with subjective "a priori" and "a posteriori" probabilities (into our model [6]).

and the limit probability distributions $q_h[=\lim_{t\to\infty} q_h(t)]$, $p_l^*[=\lim_{t\to\infty} p_l^*(t)]$ of the inventory level and total demand of costumer is known.

The conditional probabilities $f_{io}(n)$ $(n = 1, 2, 3, \ldots)$ that the level of stock reaches $s$ level after phases $n$ or remains below that, if the initial level of stock was $i$ and the order given was $S-i$ have to be determined. That is, conditional probabilities

$$f_{i0}(n) = P[\alpha_t > s(t = 1, 2, \ldots, n-1)\, \alpha_n = 0, 1, \ldots, s \,|\, \alpha_0 = i] \begin{matrix} i = 0, 1, 2, \ldots s \\ n = 0, 1, 2, \ldots \end{matrix} \tag{22}$$

are to be calculated. With the help of these relationships and the following one the expected value of the ordering period can be calculated:

$$E(T\,|\,\alpha_0 = i) = \sum_{n=1}^{\infty} n f_{i0}(n). \tag{23}$$

From this the average ordering period can be determined by the formula

$$E(T) = \sum_{i=1}^{s} x_i E[T\,|\,\alpha_0 = i] \tag{24}$$

where probabilities $x_i$ of the levels of stock in the moment of ordering can be calculated from the limit distribution $q_h$ of the level of stock and from relationship

$$x_i = \frac{q_i}{\sum_{h=0}^{s} q_h} \tag{25}$$

on the basis of conditional probability.

Let us introduce the following denotations for the determination of probabilities $f_{i0}(n)$. Let $\mathbf{h}_i$ denote row vector $\mathbf{h}_i^T = \begin{matrix} 01 \ldots i \ldots C-s \\ [00 \ldots 010 \ldots 0] \end{matrix}$ and e the column vector consisting of 1 − s only. Let $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}$ denote the matrix resulting from the omission of the first $s$-row and $s$-column of matrix $\mathbf{M}[= \sum_{l=0}^{C} p_l^* \cdot \mathbf{M}_l]$ in (18), (19). Probabilities $r_k = 1$ in matrix $\mathbf{\Gamma}_i$ if $k = S-i$ and $r_k = 0$ if $k \neq S-i$, while $r_k = 1$ in matrix $\mathbf{\Gamma}$ if $k = 0$ and $r_k = 0$ if $k \neq 0$. Furthermore, let $\mathbf{\Gamma}_0$ denote the matrix (where $r_k = 1$ if $k = 0$; 0 otherwise) of the last rows $C-s$ and the first columns $s$ omitted from stochastic matrix $\mathbf{M}$.

Let us now define the corresponding probabilities $f_{io}(n)$ $(n = 0, 1, 2\ldots)$: The probability that, after the ordered amount of materials has arrived, the store gives a repeat order at or below level s can be determined from the relationship

$$f_{io}(1) = \mathbf{h}_i^T \mathbf{\Gamma}_i \mathbf{\Gamma}_0 \mathbf{e} \ . \tag{26}$$

The probability that the order will be given in phase $n$ (for a level $i$ of stock at the time of ordering) is

$$f_{io}(n) = \mathbf{h}_i^T \mathbf{\Gamma}_i \mathbf{\Gamma}^{n-2} \mathbf{\Gamma}_0 \mathbf{e}. \tag{27}$$

Applying transformation $z$, the $z$-transformed probability $f_{io}(n)$ becomes:

$$F_{i0}(z) = f_{i0}(1)\, z + z^2 \mathbf{h}_i^T \mathbf{\Gamma}_i \sum_{n=2}^{\infty} z^{n-2} \mathbf{\Gamma}^{n-2} \mathbf{\Gamma}_0 \mathbf{e}. \tag{28}$$

With a suitable choice of $z$ — if max $|zm_{ij}| < 1$ that is, $m_{ij}$ cannot be 1 in matrix $\mathbf{\Gamma}$ (which is obvious) —, the infinite series is convergent and — by applying the above relationship (26) — it will become

$$F_{i0}(z) = \mathbf{h}_i^T \mathbf{\Gamma}_i \mathbf{\Gamma}_0 \mathbf{e} z + z^2 \mathbf{h}_i^T \mathbf{\Gamma}_i (\mathbf{I} - z\mathbf{\Gamma})^{-1} \mathbf{\Gamma}_0 \mathbf{e} \tag{29}$$

where $\mathbf{I}$ — identity matrix.

Since the Markov chain is irreducible [16], so $E(T|\alpha_0 = i) < \infty$, and $F_{i0}$ is differentiated by $z$ where $z = 1$, and, assuming that at the time of ordering the level of stock was $i$, the average ordering period is determined by the following relationship.

$$E[T \mid \alpha_0 = i] = 2 - \mathbf{h}_i^T \mathbf{\Gamma}_i \mathbf{\Gamma}_0 \mathbf{e} + \mathbf{h}_i^T \mathbf{\Gamma}_i (\mathbf{I} - \mathbf{\Gamma})^{-1} \mathbf{\Gamma} (\mathbf{I} - \mathbf{\Gamma})^{-1} \mathbf{\Gamma}_0 \mathbf{e} \tag{30}$$

After weighting the expected conditional values on the basis of relationship (24) and (25) the expected value of the ordering period is

$$E[T] = \sum_{i=1}^{s} \frac{q_i}{\sum_{h=1}^{s} q_h} [2 - \mathbf{h}_i^T \mathbf{\Gamma}_i \mathbf{\Gamma}_0 \mathbf{e} + \mathbf{h}_i^T \mathbf{\Gamma}_i (\mathbf{I} - \mathbf{\Gamma})^{-1} \mathbf{\Gamma} (\mathbf{I} - \mathbf{\Gamma})^{-1} \mathbf{\Gamma}_0 \mathbf{e}]. \tag{31}$$

Further parameters other than the independent parameters discussed above which can be important in the evaluation of stores are summed up in Table 3.

## 5. Example for Computer Investigations

The described model will be illustrated on a concrete arithmetic problem which emerged during the computer research on the system model with strategies $(s, S)$ and $(S, S)$, respectively. Behaviour of the system has been observed in 60 phases with the following parameters:

1. capacity $C$ of the store is 10 arbitrary units of materials;

2. the initial stock comprises 6 arbitrary units of materials at a probability 1;

3. the probability distribution of the output service system has been assumed to have a service system of 3 machines each of a capacity of 2 units of materials /phase.* The duration between two breakdowns of a machine is of exponential distribution with an expected value of phases $1/\lambda = 2$, repair time is also of exponential distribution, with an expected value of phases $1/\mu = 1/5$. The probability distribution of the total capacity of "$n$" machinery system consisting of machines working simultaneously but independently results from the following relationship through a simple consideration of combinatoric probability:

$$u_k(t) = v_k(t) = \sum_{k=1}^{n} \binom{n}{k/2} p_R(t)^{k/2} \left(1 - p_R(t)\right)^{n-k/2}. \tag{32}$$

In this relationship the probability that a machine will be in working order in phase $t$ is $n = 2\cdot 4$ in case of the output service system, while it is $n = 2\cdot 3$ in case of the input service system, that is, $p_R(t)$. On the basis of a formula of reliability theory the probability of a machine's being in working order in phase $t$ is determined by the following formula in case of exponentially distributive periods of working order and repair:

$$p_R(t) = \frac{\mu + \lambda \cdot e^{-(\lambda+\mu)\cdot t}}{\mu + \lambda}. \tag{33}$$

According to formula $(P_R = \mu/\lambda + \mu)$; in a permanent state the probability distribution vector of the capacity of the service system is $\mathbf{v} = [0.0008, 0.0068, 0{,}0834, 0.2245, 0.1578, 0.000, 0.5259, 0.000, 0.000, 0.000, 0.000]$;

4. distribution of the capacity of the input service system can be obtained similarly to that of the output service system but we have assumed to have 3 handling machines instead of 4. In this case the probability distribution of the capacity of the input system in a permanent state results from the calculations above. $\mathbf{u} = [0.0001, 0.0008, 0.0142, 0.0820, 0.2336, 0.000, 0.1912, 0.000, 0.4781, 0.000, 0.000]$;

5. demands arrive in every phase according to the Poisson distribution of 2 expected values.

* In a case more complex than the one described in our example, where the capacity of the individual machines is a random variable, the probability distribution of the capacity of the output machinery system can be determined according to [10].

**Table 4** Output list of computer jobs

| PHASE: | (3) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| DEMAND | 0.0734 | 0.1847 | 0.2218 | 0.1910 | 0.1367 | 0.0862 | 0.0497 | 0.0260 | 0.0133 | 0.0062 | 0.0045 |
| WAITING DEMAND | 0.3102 | 0.1746 | 0.1715 | 0.1350 | 0.0914 | 0.0556 | 0.0313 | 0.0165 | 0.0081 | 0.0037 | 0.0022 |
| INPUT | 0.9543 | 0.0088 | 0.0202 | [illegible] | 0.0047 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| INPUT QUEUE | 0.9543 | 0.0088 | 0.0202 | [illegible] | 0.0047 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| INPUT CAPACITY | 0.0001 | 0.0008 | 0.0142 | [illegible] | 0.2336 | [illegible] | 0.1912 | 0.0000 | 0.4781 | 0.0000 | 0.0000 |
| OUTPUT CAPACITY | 0.0003 | 0.0080 | 0.0834 | [illegible] | 0.1578 | [illegible] | 0.5259 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| STOCK LEVEL | 0.5098 | 0.1567 | 0.1264 | [illegible] | 0.0463 | 0.0252 | 0.0170 | 0.0135 | 0.0103 | 0.0065 | 0.0050 |

M(3) =

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.9836 | 0.0087 | 0.0051 | 0.0020 | [illegible] | 0.0001 | [illegible] | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8977 | 0.0859 | 0.0087 | 0.0051 | [illegible] | 0.0005 | [illegible] | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.7079 | 0.1899 | 0.0859 | 0.0087 | 0.0051 | 0.0020 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 0.4514 | 0.2565 | 0.1899 | 0.0859 | 0.0087 | 0.0051 | 0.0020 | 0.0005 | 0.0001 | 0.0000 | 0.0000 |
| 0.2125 | 0.2385 | 0.2565 | 0.1899 | 0.0859 | 0.0087 | 0.0051 | 0.0020 | 0.0005 | 0.0001 | 0.0000 |
| 0.0931 | 0.1188 | 0.2385 | 0.2565 | 0.1899 | 0.0859 | 0.0087 | 0.0051 | 0.0020 | 0.0005 | 0.0000 |
| [illegible] | 0.0437 | 0.1188 | 0.2385 | 0.2565 | 0.1899 | 0.0859 | 0.0087 | 0.0051 | 0.0020 | 0.0006 |
| 0.0000 | 0.0503 | 0.0437 | 0.1188 | 0.2385 | 0.2565 | 0.1899 | 0.0859 | 0.0087 | 0.0051 | 0.0026 |
| 0.0000 | 0.0000 | 0.0503 | 0.0437 | 0.1188 | 0.2385 | 0.2565 | 0.1899 | 0.0859 | 0.0087 | 0.0077 |
| 0.0000 | 0.0000 | 0.0000 | 0.0503 | 0.0437 | 0.1188 | 0.2385 | 0.2565 | 0.1899 | 0.0859 | 0.0164 |
| 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0503 | 0.0437 | 0.1188 | 0.2385 | 0.2565 | 0.1899 | 0.1023 |

N(3) =

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.9579 | 0.0151 | 0.0183 | 0.0065 | 0.0011 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.8874 | 0.0161 | 0.0136 | [illegible] | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.7099 | 0.0132 | 0.0065 | 0.0013 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.4581 | 0.0065 | 0.0028 | [illegible] | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.2165 | 0.0025 | 0.0019 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.1229 | 0.0957 | 0.0024 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.1229 | 0.0453 | [illegible] | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.1229 | 0.0453 | 0.0528 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.1229 | 0.0453 | 0.0528 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.1229 | 0.0453 | 0.0528 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0791 | 0.1895 | 0.2631 | 0.2472 | 0.1229 | 0.0453 | 0.0528 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

| | expected value | variance | entropy |
|---|---|---|---|
| DEMAND | 2.872 | 1.345 | .922 |
| WAITING DEMAND | 1.990 | 1.93[illegible] | [illegible] |
| INPUT | 0.107 | 0.532 | 0.358 |
| INPUT QUEUE | 0.107 | 0.532 | 0.358 |
| INPUT CAPACITY | 6.182 | 1.950 | 1.848 |
| OUTPUT CAPACITY | 4.636 | 1.527 | 1.748 |
| STOCK LEVEL | 1.358 | 1.979 | 2.287 |
| OUTPUT | 0.884 | 1.155 | 1.848 |

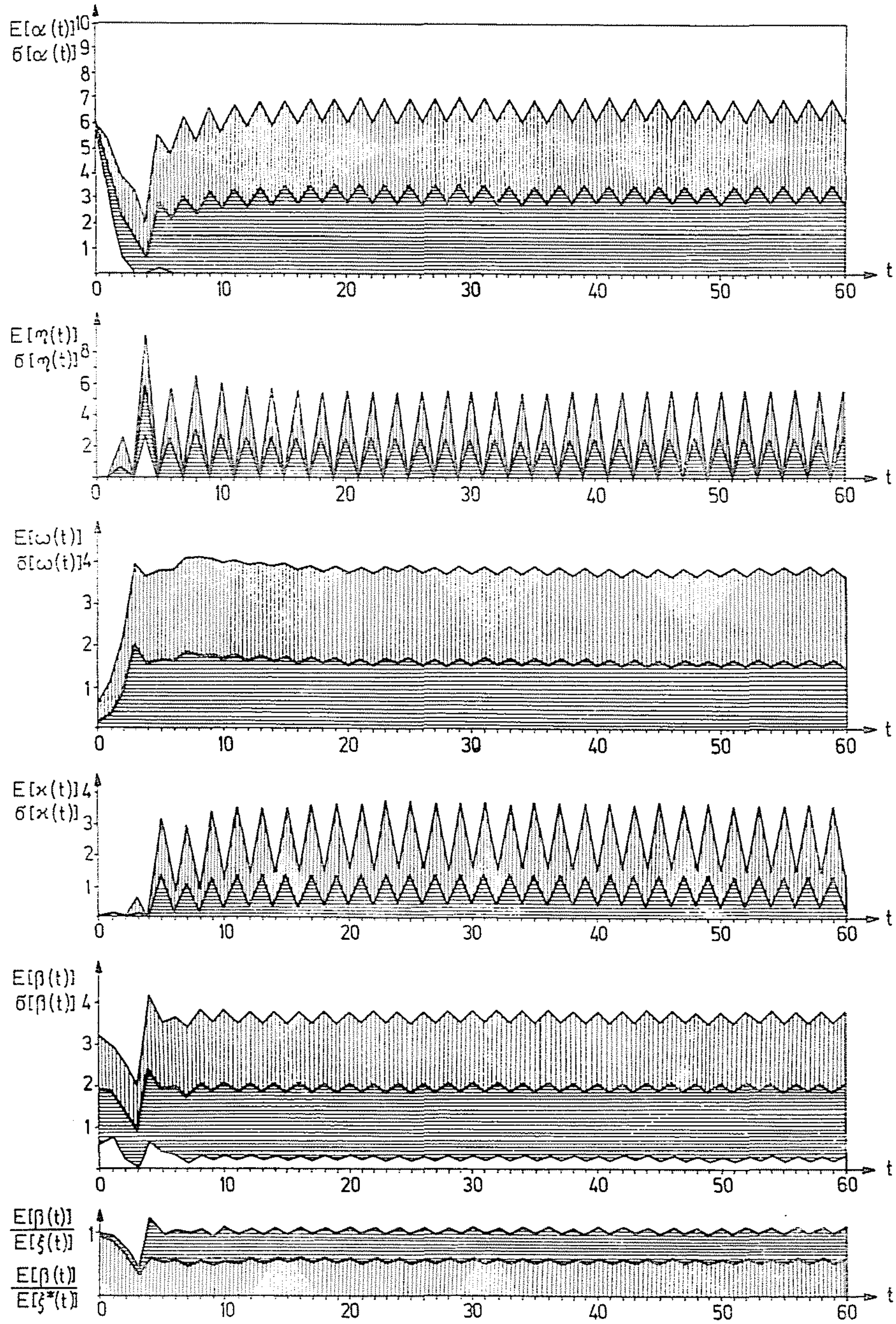


*Fig. 6.* Diagram showing the changes in storage system (of (s, S) type) behaviour

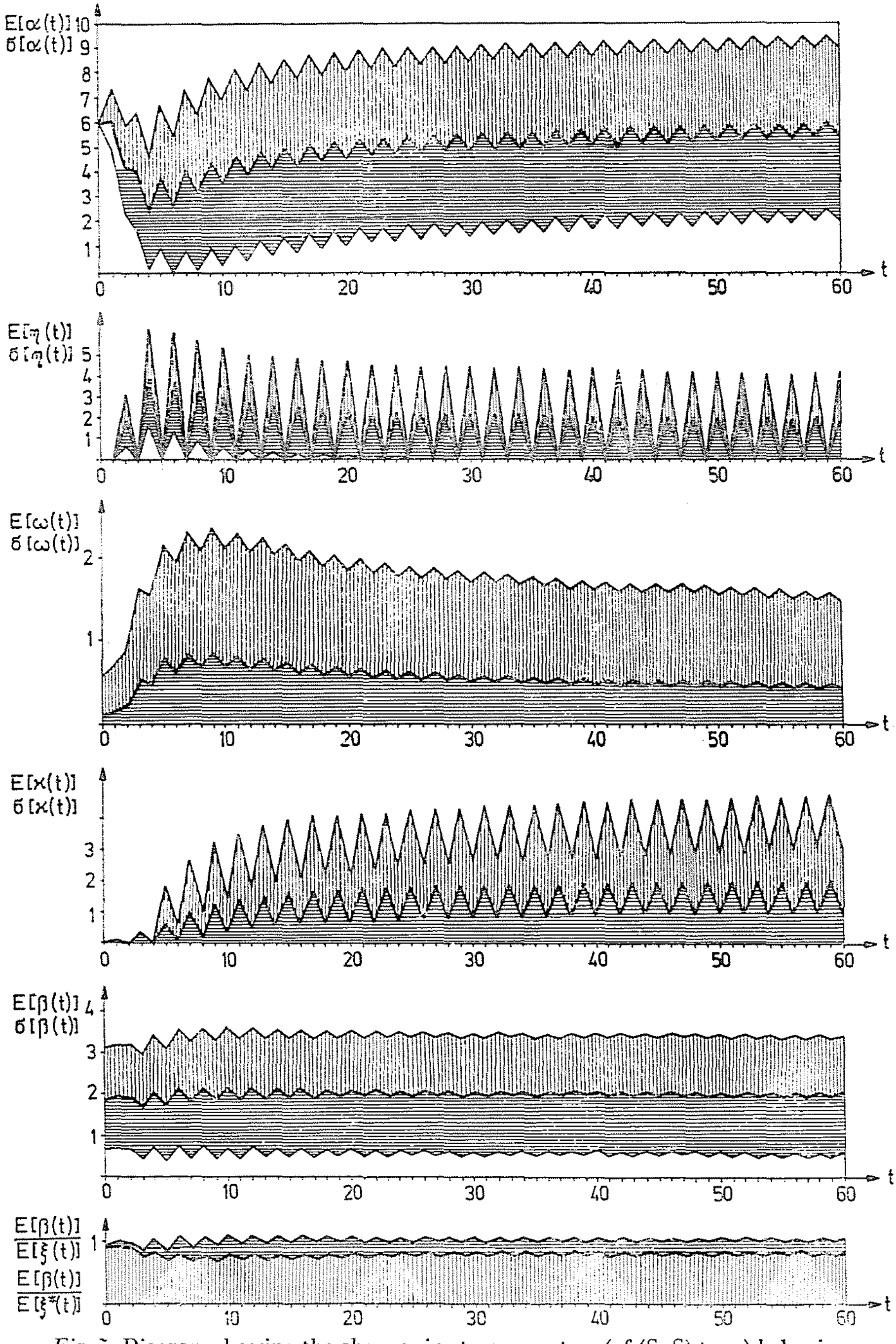


*Fig. 7.* Diagram showing the changes in storage system (of (S, S) type) behaviour

6. orders are "given" by the system at the "end" of every second phase (a) according to strategy $(s, S)$ $s = 3$; $S = 8$; (b)(according to strategy $(S, S)$ $S = 8$; $T = 2$). The ordered amount of materials arrives at "the beginning of" the next phase.

Table 4 (for illustration of computer research) shows the transition and output probability matrices, the probability distributions of the most important system parameters, their standard deviation and expected values on the output list of computer jobs. Figs. 6 and 7 illustrate the changes in the parameter values of the behaviour of the system with strategies $(s, S)$ and $(S, S)$, respectively, that is, the expected value and the variance of the level of stock of the repeat-order amount of materials, of the length of queues of arrivals and demands, of the amount of output materials (demands satisfied), and of degrees of functional reliability vs. time. (Variance is marked by shading in the diagram.)


## Summary

The complex model (algorithm) constructed for the investigation of storage system behaviour is controlled by a stochastic flow of demands, has a memory (queues) and feedback (inventory control). It can be applied for a realistic description of the "life" of the store with respect to the influence of the internal processes of the store and the capacity of its service systems if sufficient practical information is available. The application of the model (resulting in an extension of inventory control theory and storage-technological investigations) can be particularly significant in investigating, designing and controlling automated overhead high — storage systems. Furthermore, it can be used in identifying the system through observation and measurements, in lack of "a priori" information.


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