INTERVAL ESTIMATION PROCEDURES FOR EVALUATING FATIGUE DATA BASED ON THE WEIBULL DISTRIBUTION*

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1. Introduction

All statistical design methods rely on a statistical approach of expected stresses and material characteristics. Results are probabilistic, e.g. useful life at a given probability.

Design load values and material characteristics are determined experimentally, by statistical methods. Irrespective of the "quality" of the design method, results are much dependent on the accuracy or correct interpretation of design data.

In the following, some aspects of material testing related to fatigue design will be considered.

The fatigue characteristics of materials or parts are generally described by the Woehler curve. In individual cases a single result of program load testing, random load or service load fatigue testing series will be relied on. These tests have the common feature of consisting of observations made with some parameters kept constant, and the obtained sample is applied to predict e.g. the permissible stresses, the expected life etc. Deductions are based generally on small samples stressing the applied statistical method and the correct interpretation of results.

In the following, some mathematical methods likely of an exact interpretation and practical evaluation of results from small samples ($n < 50$ to 100) will be presented.

2. General considerations

In processing the test results — considered as a sample taken at random from a population characterized by some distribution function — parameters of the assumed distribution function are estimated to draw further conclusions.

2.1 The type of the distribution function to be assumed cannot be unam-

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biguously determined from the available, generally small samples. Thus, earlier tests and theoretical considerations should be involved.

Based on large-sample fitting tests [14] and some theoretical considerations [4], in general, the Weibull distribution (type III extreme value distribution) is accepted.

In certain domains the lognormal distribution has been found to truly fit test data. Simple of handling, this type of distribution is widely accepted [2] [3].

The selection of the initial distribution function is of importance from two aspects:

2.1.1 Possibility of extrapolation for low probabilities or ranges not covered by the sample (e.g. probability range 0 to 0.1; 0.90 to 1 upon a sample of 10 elements).

2.1.2 In case of small samples, the features of statistics for estimating parameters (distribution, variance etc.) are much dependent on the selected model. (This fact is less important for large samples subject to the asymptotic theory.)

2.2 The parameters of the applied model are estimated by statistical functions of the sample elements. These functions of random variables are random variables themselves.

Point estimation procedures assign a single value to the estimated parameter. They contain, however, no direct information on the degree of their fluctuation, or on the statistical reliability of the estimation. Neither the effect of sample size can directly be appreciated.

Rather than a single value, interval estimations give an interval (or range) for the parameter that is within the range at a given probability. A one-sided lower confidence limit gives a lower estimation at a prescribed probability. The interval width is characteristic of the estimation accuracy, directly concludes on the reliability of the statement based on the estimation, or on its uncertainty. Effect of increasing the sample size can exactly be evaluated.

Interval estimation procedures of the lognormal model based on the Gaussian distribution are easy to obtain [15]. In the following the case of the Weibull distribution will be discussed.

3. Estimation procedures based on the Weibull model

The distribution function of the random variable ξ_i $(i = 1 \dots n)$ of Weibull distribution can be written as:

$$P(\xi < x) = F(x) = 1 - \exp\left\{-\left(\frac{x - x_0}{\delta}\right)^b\right\} \quad x \geq x_0 \tag{1}$$

here: $b > 0$ shape parameter, $\delta > 0$ scale parameter, $x_0 \geq 0$ location parameter.

In the most general case, all three parameters are estimated from the sample. In case of small samples, the method of moments [1] is rather uncertain, because of the estimation of the slope. Maximum likelihood equations are only regular for $b > 2$, hence they are impractical as a rule.

The most familiar method consists in estimating parameters by graphic plotting ([3], [4]). In this method, some plotting position p_i is assigned to the ordered sample elements $x_{i,n}$ (e.g. $p_i = i/(n+1)$ is the expected value of $F(x_{i,n})$ [13], its mediane [5], or $p_i = \left(i - \frac{1}{2}\right)\Big/n$ etc.) Plotting the correlated values on a Gumbel probability paper [4] or — after proper transformation — in a log-log reference system [16], the smoothed curve, fitted to the points gives the estimation of the distribution sought for. If the points fit a straight line, $x_0 = 0$ can be accepted as estimation for parameter x_0. Estimations for b and δ are given by corresponding parameters of the smoothed line. For a poor fitting to the striaght line, a fair estimation of parameter x_0, plotting $\xi_i - x_0'$ values yields a straight line if x_0' is fairly estimated. The straight line fitted to the data is essentially a least squares estimation, giving a result free of subjective errors upon analytic calculation [14].

In case of small samples, estimations resulting from these methods cannot be exactly appreciated. The distributions, variances of the estimations can only be determined for large samples, using asymptotic theory.

Irrespective of the applied procedure, the examination of the distribution of ordered sample elements may be instructive of the possible accuracy [13]. On this basis, L. G. Johnson examined the reliability of estimations for the two-parameter model. This, however, permits conclusions on the test range alone — if parameter b is known.

Distributions of other statistics can, hovever, be determined also for small samples, permitting conclusions on the reliability of results.

3.1 Provided $x_0 = 0$ is acceptable on the basis of the sample — or of preliminary considerations — (e.g. graphic analysis), maximum likelihood estimations are quite efficient, related to the Cramer Rao lower bound, even for small samples. They are asymptotically unbiased; with the aid of the unbiasing factors, determined by simulation, they can be taken unbiased even for small samples [10].

These estimations result from the iterative solution of maximum likelihood equations

$$\frac{n}{\hat{b}} - n\frac{\sum_{i=1}^{n} x_i^{\hat{b}} \ln x_i}{\sum_{i=1}^{n} x_i^{\hat{b}}} + \sum_{i=1}^{n} \ln x_i = 0 \tag{2}$$

$$\hat{\delta} = \left(\sum_{i=1}^{n} x_i^{\hat{b}}/n\right)^{1/b}$$

where x_i $(i = 1 \ldots n)$ is a sample of n elements with the distribution function (1), $\hat{b}$ and $\hat{\delta}$ are parameter estimations.

The distribution of $\hat{b}/b$ can be demonstrated to be the same as that of $\hat{b}_{11}$, and that of $\hat{b} \ln (\hat{\delta}/\delta)$ as that of $\hat{b}_{11} \ln \hat{\delta}_{11}$ where $\hat{b}_{11}$ and $\hat{\delta}_{11}$ are maximum likelihood estimations of a random variable of parameters $b = 1$, $\delta = 1$, and distribution (1). Distribution of $\hat{b}_{11}$ and $\hat{\delta}_{11}$ can be determined by Monte Carlo simulation. Hence, confidence interval at $1 - \varepsilon$ level can be given for b and δ, with the aid of the percentage points of the distributions of $\hat{b}_{11}$ and $\hat{\delta}_{11}$:

$$P(b_1 < b < b_2) = 1 - \varepsilon$$
$$P(\delta_1 < \delta < \delta_2) = 1 - \varepsilon \tag{4}$$

where b_i and δ_i $(i = 1, 2)$, as functions of n, can be calculated, based on the tables of the distribution of $\hat{b}_{11}$ and $\hat{\delta}_{11}$ [11]. Similarly, one-sided confidence bounds can be given.

3.2 Based on theoretical and empirical studies, the best linear invariant and the best linear unbiased estimations have certain good properties, which are similar to that of the maximum likelihood estimations. They are easy to obtain, by the transformation of $y_i = \ln x_i$, in the following form: $\beta = \sum_{i=1}^{n} a_{i,n} y_i$, $\Theta = \sum_{i=1}^{n} b_{i,n} y_i$, where y_i are of the type I extreme value distribution, with parameters $\beta = 1/b$, $\Theta = \ln \delta$. Optimum weights $a_{i,n}$ $b_{i,n}$ $i = 1 \ldots n$, can be obtained from tables [8]. Determining by simulation the distribution of certain functions of the parameters [9], confidence intervals or confidence bounds similar to those in item 3.1 can be given for parameter estimation.

For the probability of survival $R(x)$, defined by $R(x) = 1 - F(x)$, to a given x_ε, lower confidence bound on $R(x_\varepsilon)$ at a given probability $1 - \varepsilon$ can directly be calculated [9] [12]. The above considerations will be presented on the following examples.

The results of a life test of five specimens [5] plotted on Gumbel probability paper, give a good fit to a straight line, so $x_0 = 0$ can be accepted. The maximum likelihood estimations of the parameters are $\hat{b} = 1.95$, $\hat{\delta} = 185$; with the unbiasing factor for b, $\hat{b} = 1.31$. The 90% level confidence intervals for the parameters are:

$$(0.705 < b < 2.87) \quad \text{and} \quad (105 < \delta < 349)\,.$$

The confidences for the percentage points, given by the above equations, are presented in Fig. 1. From practical point of view we have given also the one-sided lower bound, belonging to the 90% level lower confidence bounds of the parameters.

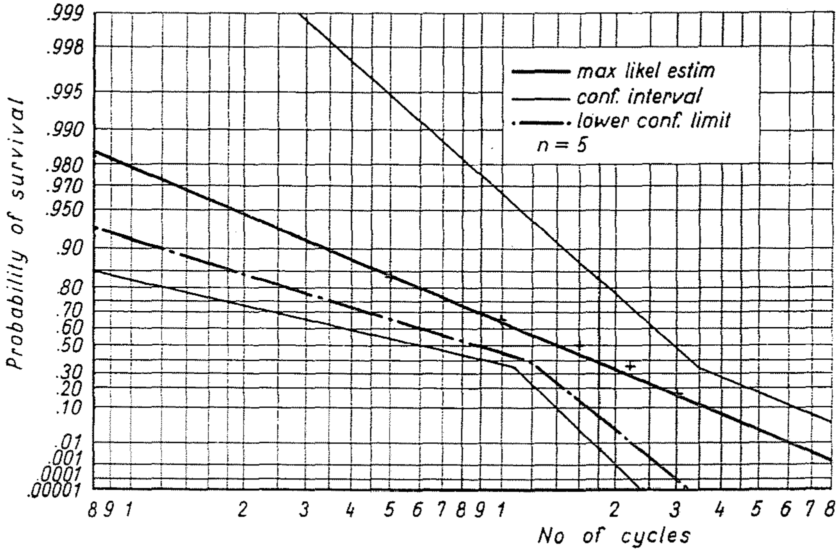


Fig. 1

For a life test of ten specimens [5], the confidence intervals are much restricted (Fig. 2).

The life test data of 23 ball bearings [6] also plot reasonably well as a straight line, on Gumbel probability paper, so a two-parameter model can be assumed ($x_0 = 0$). The maximum likelihood estimates and the confidence intervals are presented in Fig. 3.

3.3 If assumption $x_0 = 0$ is opposed by preliminary considerations or graphic analysis, procedures in 3.1 and 3.2 are useless (provided x_0 is known or pre-estimated). No method is known for calculating exact confidence intervals for small samples.

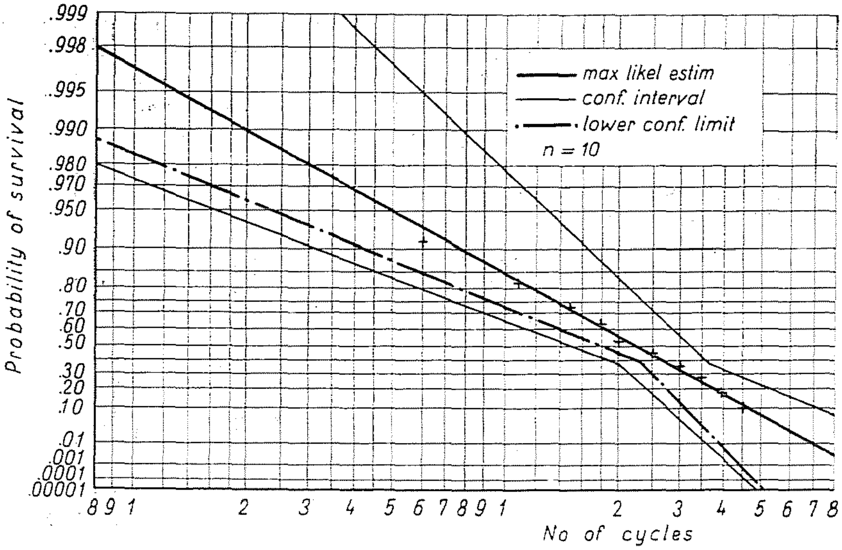
From practical aspects, however, the three-parameter model is needed exactly for describing the range of low failure probability, decisively affected by parameter x_0; for $x < x_0$, $F(x) = 0$.

But a confidence interval or lower confidence bound can be given for the location parameter x_0 after appropriate transformation of the sample elements.

Let us consider a sample of n elements ξ_i, $i = 1 \ldots n$, taken at random from a population of distribution (1), arranged in ascending order. Then the common density of the random variables

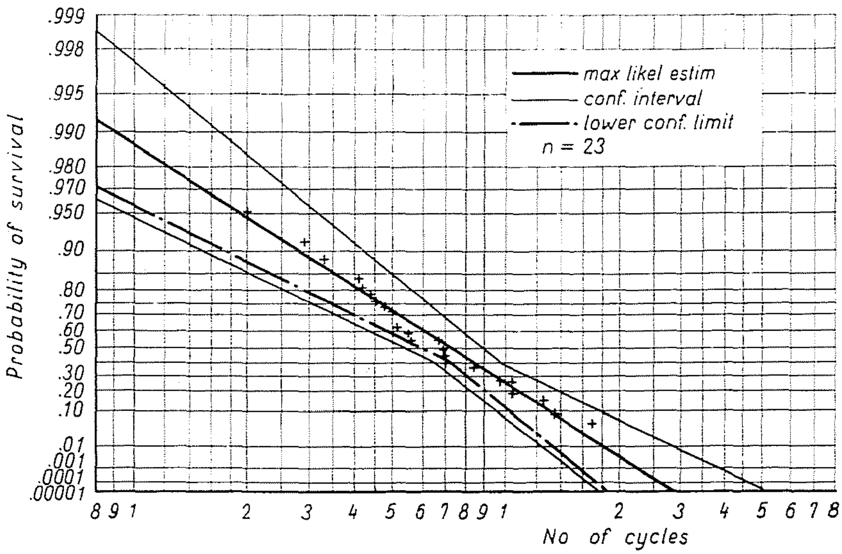
$$V_i = \left(\frac{x_i - x_0}{x_1 - x_0}\right)^b \qquad i = 1 \ldots m \leq n$$
$$V_1 = 1 < V_2 < \ldots < V_m \qquad m \leq n \tag{5}$$

can be demonstrated to be independent of δ [10].



max likel estim
conf. interval
lower conf. limit
n = 10
Probability of survival
.999
.998
.995
.990
.980
.970
.950
.90
.80
.70
.60
.50
.30
.20
.10
.01
.001
.0001
.00001
8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8
No of cycles

Fig. 2



max likel estim
conf. interval
lower conf. limit
n = 23
Probability of survival
.999
.998
.995
.990
.980
.970
.950
.90
.80
.70
.60
.50
.30
.20
.10
.01
.001
.0001
.00001
8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8
No of cycles

Fig. 3

Taking the statistic

$$U = \frac{1}{n(m-1)} \sum_{i=1}^{m} (n-i+1)(V_i - V_{i-1}) \tag{6}$$

then, if x_0 is the true location parameter, (6), as the ratio of two independent chi square variables, each divided by its number of degrees of freedom is of Snedecor's F-distribution with the parameters $2(m-1)$, 2, [10], where m indicates the type two censoring from above.

Thus, taking U equal to $F_{1-\varepsilon/2}$ or $F_{\varepsilon/2}$, the iterative solution of the resulting equations gives confidence interval for x_0 at level $1-\varepsilon$, where $F_{1-\varepsilon/2}$ and $F_{\varepsilon/2}$ are the upper and lower $\varepsilon/2$ percentage points, respectively, of the Snedecor distribution of $2\,(m-1)$, 2 degrees of freedom [10].

In a similar way, equation $U = F_\varepsilon(2[m-1], 2)$ gives an $1-\varepsilon$ level lower confidence bound for the location parameter x_0 if F_ε is the lower ε point of the Snedecor distribution of $2(m-1)$, 2 degrees of freedom.

Percentage points of distribution F may be taken from tables or calculated by the relationship [10]:

$$F_{\varepsilon(2[m-1],2)} = \frac{1}{m-1} \; \frac{\varepsilon^{1/m-1}}{1-\varepsilon^{1/m-1}} \,. \tag{7}$$

The statistic (6) gives also a statistical test for testing the hypothesis $H_0 : x_0 \geqslant x_0'$ or $H_0' : x_0 \leqslant x_0'$. In case of simple alternatives of practical importance, the power of the test is rather high [10].

Thus, the presented procedure gives a confidence interval for x_0 if b is known at least approximately. Otherwise it can be replaced by its value estimated from the sample, giving a conditional confidence interval.

4. Application in case of a multistage fatigue test

Application of procedures described in item 3.3 will be presented on the results of a large-sample fatigue test. Test results refer to notched specimens made of steel Ck 35 (DIN), tested at an extraordinary care [7]. In the range $\sigma = 38.5$ to (0.5) to 32, none of the tested 20 specimens at each level, (24 at $\sigma = 32$ kp/mm^2) endured 10^7 cycles. Thus, the fatigue range can be ended at 32 kp/mm^2. Our computation results refer to this range (Fig 4.) [7].

From the plot at each level, the assumption $x_0 = 0$ seems to be inadequate, especially for lower stresses (Fig. 5). There are also deviations from the lognormal distribution.

As a comparison, maximum likelihood estimations have been made of parameters and percentage points, assuming $x_0 = 0$ as trivial lower bound

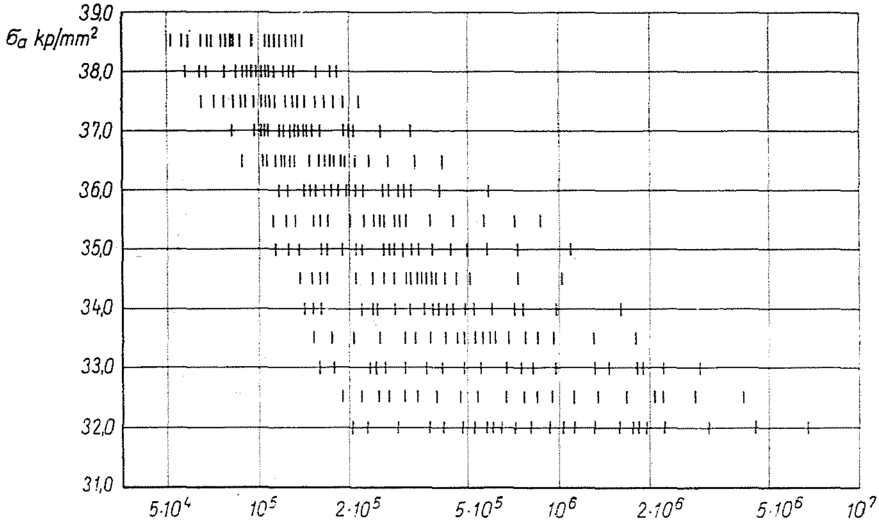


Fig. 4

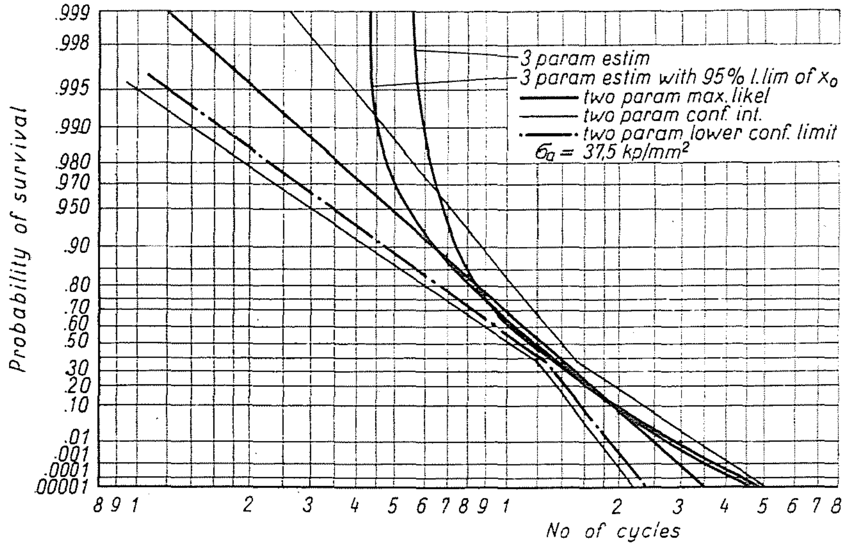


Fig. 5

of the location parameter. Values obtained for 1%, 10%, 50% points are shown in Fig. 7. As a comparison, lognormal model results have also been plotted [7]. Agreement can be stated to be statisfactory in the 10 to 50% range. Because of the trivial lower estimation of x_0, confidence intervals for the individual percentage points give no practically evaluable results. For 50% failure probabilities, however, the lower estimates, based on the 90%

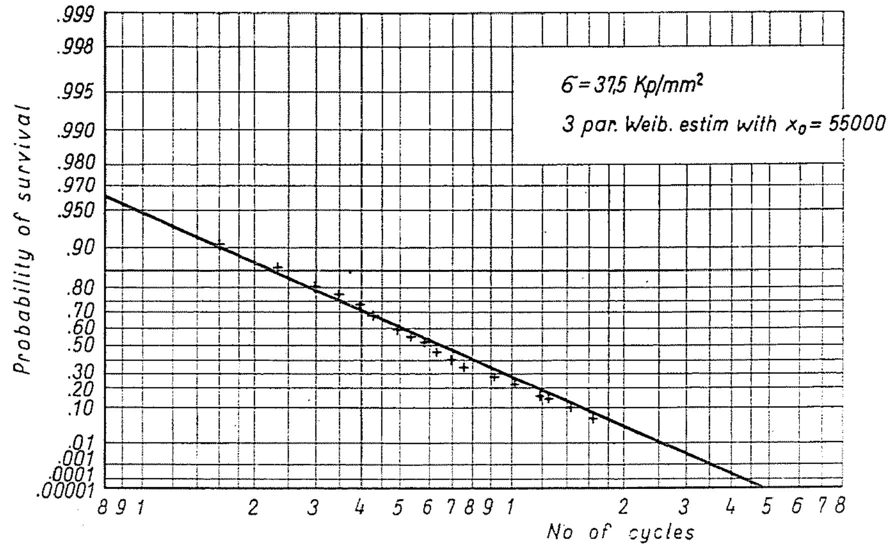


Fig. 6

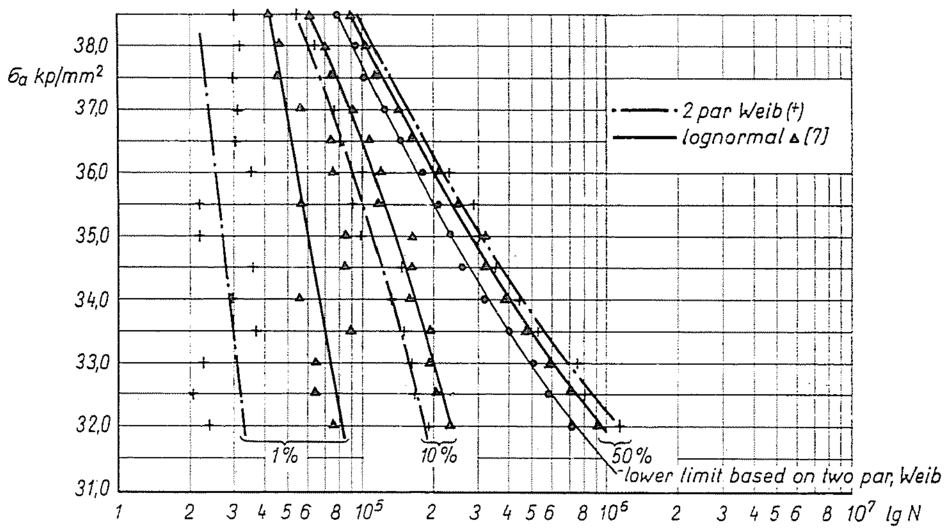


Fig. 7

max. likel. one-sided lower limits of the parameters, have also been indicated (Fig. 7).

From graphic analysis results, evaluation in the failure range has been made with the assumption $x_0 > 0$.

Analytic iteration has been applied for parameter estimations of the three-parameter model. Three-parameter evaluation resulted in a rather fair

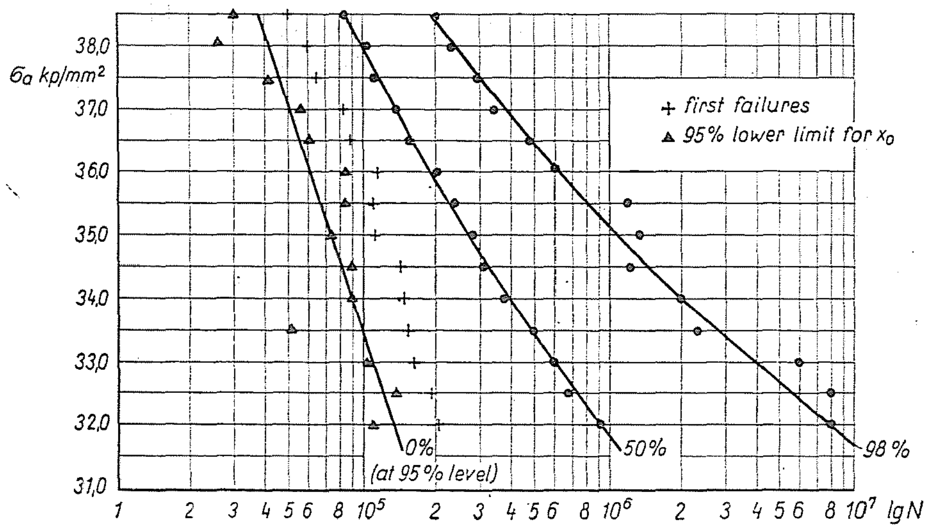


Fig. 8

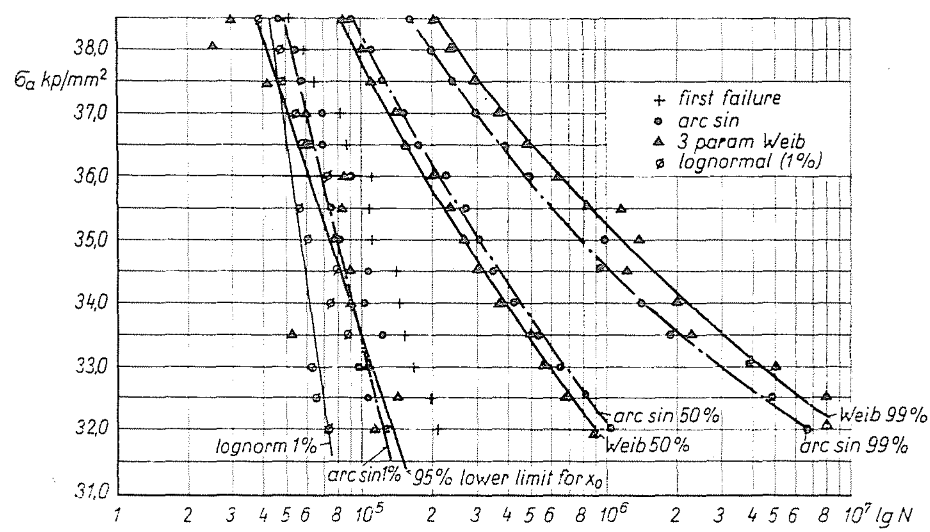


Fig. 9

fitting at each level (Fig. 6). From the results, making use of the estimation of b, 95% lower confidence bound has been obtained (Fig. 8).

Thus, if x_0' is the lower confidence limit, the obtained points are interpreted as:

$$P(x_0 \geq x_0' \mid b = \hat{b}) = 0.95 \tag{8}$$

where x_0 is the true location parameter.

As a comparison, 1% failure probability estimations have been plotted according to the modified arc sin transformation suggested by MAENNING [7]. Here the probability estimation is arc sin $\sqrt{(i - 0{,}8)/(n + 1)}$ found heuristically (Fig. 9).

For sake of comparison, the 1% failure probability points have been plotted for the lognormal case, too (Fig. 9).

The calculations were performed on an Odra 1204 computer, programmed in Algol language. The calculations required only a few seconds for each level.

5. Conclusions

1. Reliability of estimated values can also be concluded on in case of the Weibull distribution, by means of parameter estimations given by statistical functions of known distribution, or determinable by simulation. In case of two-parameter models, reliability of the obtained values ($x_0 = 0$) can exactly be determined. Accordingly, estimation of percentage points can be evaluated and effect of increasing the sample size determined. Knowledge of the confidence intervals is also instructive for determining the "safety factors" sometimes required.

2. In case of a three-parameter model, confidence interval for estimating the location parameter x_0 can be given by means of statistic (6), making use of an at least approximate value of b. Replacing b by its value estimated from the sample leads to exact conditional confidence limits for the assumed b values.

3. Effect of increasing the sample size can be directly read off Figs 1 to 3. Assessment of low or high failure probabilities is rather uncertain, especially for small samples.

4. From Fig. 7 showing 50% failure probability lower confidence limits, reliability of estimations is seen to decrease with lower b values (i.e. with decreasing stresses). Hence, for a given sample, at lower stresses it is advisable to test more specimens. Numerical distribution can be made according to parameter pre-estimations.

5. The lower confidence bounds for the location parameter x_0 show a fair agreement with 1% failure probabilities indicated by MAENNING. This is rather interesting since the two values are concluded on from quite different theories. Although they differ by interpretation, they can be considered similarly, as design values. (At higher stresses, however, the differences are greater. But taking into account the first failures, the 1% values given by arc sin transformation are rather high, Fig. 9.)

6. In the 10 to 50% range, both the two-parameter Weibull, and the lognormal distribution yield fairly agreeing results. Effect of the selected model becomes decisive for low values.

7. The lower confidence bound according to (6) of parameter x_0 means the limitation of the failure range at a given probability. Thereby the plane $\sigma - N$ can be divided into a failure-free zone and a failure zone at a given probability.

8. In lack of a preliminary information, an estimation according to statistic (6) is influenced by the uncertainty of the estimate of b. Hence, it can only be applied for larger samples.

9. For constructing a Woehler curve by the presented methods, some indication may be obtained for the values of the curve. For each failure probability, the fitting to the lower confidence bounds yields lower values in case of small smaples because of the greater intervals. These taken into consideration, effect of the sample size can directly be described. For a larger sample, relatively higher cycles are obtained as lower bounds at the same reliability.

10. The presented methods can widely be used in certain fields of fatigue testing and load analysis (e.g. extreme loads).

Summary

Based on the fact that life test data are usually rather scattered, fatigue characteristics of the material are better determined by statistical methods. The estimation of the parameters of the assumed distribution as well as estimation of life test data involve a certain degree of fluctuation. Interval estimation procedures are quite adequate for the evaluation of fatigue data since they lead to definite conclusions on the reliability or uncertainty of the statement arrived at by these procedures. Methods for obtaining confidence bounds based on the Weibull model are presented. Numerical examples based on life test data are given and the results are discussed.

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