A STOCHASTIC MODEL FOR THE DESCRIPTION AND EVALUATION OF STORAGE SYSTEM BEHAVIOUR

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Introduction

In this paper, the behaviour of storage systems will be described from a new viewpoint. The characteristics of storage system behaviour are jointly determined by (1) environmental influences, (2) the function and (3) the capacity of the store, (4) the reliability and (5) the through-put capacity of its service system and (6) the input and release rules. The comprehensive stochastic model described here algorithmizes the description and evaluation of the behaviour system with a view to the above mentioned factors. The advantage of the complex reliability-oriented model over the rather static models of inventory control theory is that on the latter the fundamental technical and reliability characteristics are given a more prominent role in the description of the behaviour of storage systems. The model, the algorithm and the computer simulations based on them facilitate to design storage systems, development and research of particular systems and preparatory phases of investment decisions.

1. The role of storage systems in the flow of materials

The flow of materials between the supplier and customer systems is induced by the demands of the latter system which, in turn, possesses stochastic characteristics. The demands flow from the customer system to the supplier system inducing the flow of materials in the opposite direction (Fig. 1).

Storage systems have a function of balancing and controlling in the flow of materials between the customer and supplier systems. Balancing and controlling are characteristic features of storage systems functioning without and with inventory control, respectively.

If both the emergence of demand and the flow of materials from the supplier system are deterministic and without phase lag the marginal case of a storage system results where the input quantity is always equal to that of the released material. The best illustration of this case can be obtained by ima-

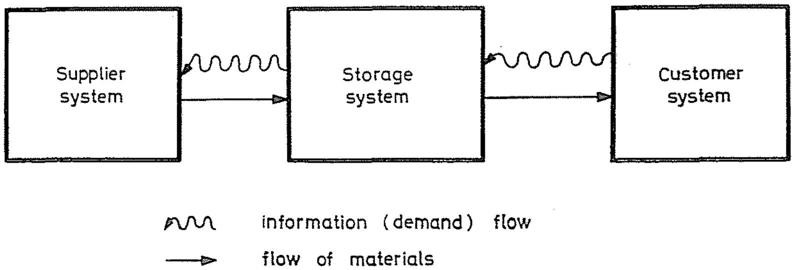

Fig. 1. Processes affecting storage systems

gining the flow of some material in a "tube". In case of phase lag an extra dam capacity has to be added to the "tube" in case of both deterministic and stochastic flows, in order to safely meet the demands. In this case the storage system has a balancing function.

In case of stochastic flows without phase lag, the existence of the storage system is justified by safety reasons.

2. The systems engineering interpretation of the behaviour, conduct and functioning of the store

The behaviour of the storage system (that is, the change of its parameters relating to the function of time experienced by the environment) is determined by the changes in the information flow from the customer system and by the changes in the flow of materials from the supplier system, and, also by the changes in the service ability conditions of the store itself.

The conduct of the storage system is manifest in the rules which determine the immediate connections between the store and its environment. If the store operates on an inventory control basis then the inventory control strategy means the conduct the store displays towards the supplier system. Otherwise passive conduct of the system as to the input environment can be spoken of. (For instance, the report that "the store is full" is a rule of conduct in this meaning.) The connection between the storage and the customer systems is characterized by the so called release conduct. The latter means the rules on the basis of which the system, prompted by the demands arriving, releases the materials "according to its own abilities" (e.g. priority rules, adaptive release rules, etc.). The expression "according to its own abilities" indicates that the satisfaction of demands is limited by the technological and reliability parameters and the level of storage at the moment.

From the point of view of systems engineering *the (internal) functioning of the storage system* can be interpreted as follows.

The input flow of materials arrives at its appointed place in the store (according to some plan of the indoor distribution of materials) through the service system. The output flow of materials governed by the flow of demands leaves the store through the service system (according to a certain output strategy).

The input and output flows of materials are limited by the throughput capacity of the service system. The throughput capacity of the service system depends on the volume and reliability of the equipment capacity as well as on the applied strategies of output and of the indoor distribution of materials. Therefore simulating the throughput ability of the different equipment systems it is useful to consider, with the help of different methods, the throughput ability, the level of capacity as a random variable.

3. Principles of investigating the behaviour of storage systems

The storage system can be considered as a system to be mathematically modelled by stochastic processes since the input and output flow and the internal functioning of the store is random.

In the systems engineering model describing the behaviour of the store *the input signals* of the system from the customer's point of view correspond to the set of information of demands arriving from the customer system, while its *output signals* correspond to the set of information dispatched to the customer system by the store. The state of the system is characterized by the inventory level of the store (Fig. 2).

In this case simulation of the storage system is oriented towards reliability problems, so the additional characteristics involved in the model may be considered as sources of noise. Three different sources of noise can be distinguished in the model of the system. The external source of noise is the random fluctuation of the input material flow, while the two interior sources correspond to the random fluctuation of the capacity of the input and the output service systems. There has been two type of research concerning storage systems.

- Research concerning store technology concentrate on the machine units of the system, on the functioning of and the connections between its sub-systems and on the internal processes of the store. They do not consider the store as an organic whole, and they fail to investigate the interrelations between the system and the environment.
- Inventory control strategies and dam theories treat the storage system as a "black box". In describing its behaviour and operation they concentrate on the cost-factors involved and neglect as a rule, the factors of technology and those of the flow of materials.

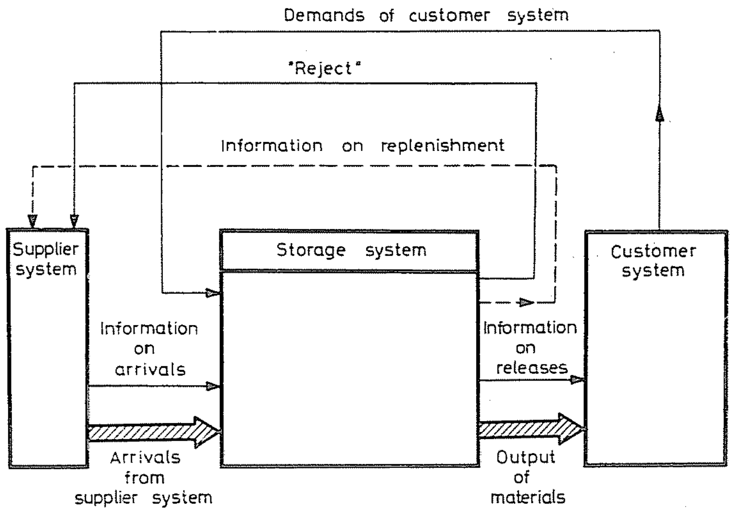

Fig. 2. Storage system in its relation to the flow of materials and information

The aim of the stochastic model described in our paper is "to look into the black box" by unifying the inventory control theory and the description of the internal processes of the store, that is, to give a description in relation to time of the store and of its interrelations with the environment, where its basic technological and reliability parameters and, also, its rules of conduct are taken into account and the store is considered as an organic whole. This description would make the behaviour of the store suitable for evaluation.

4. General model for the description of the behaviour of storage systems

4.1 *Conditions determining the behaviour of storage systems*

With respect to the described model, the characteristics of the "behaviour" of a storage system can be summarized as follows.

4.1.1. The behaviour in relation to. time of the system is investigated in discrete periods.

4.1.2. In every period the supplier system releases material towards the storage system.

4.1.3. In each period the customer system releases demands to the supplier system.

4.1.4. The storage system is assumed to receive the materials that have arrived from the supplier system "according to its own abilities" and to satisfy the demands that arrive within the given period if it is made possible by the level of storage and the capacity. (This condition can be met by a suitable choice of periods.)

4.1.5. The storage system does not release more material than is required by the demands, and rejects the quantity of arriving materials in excess of its capacity.

4.1.6. The behaviour of the storage system involves the level of storage and the demand satisfied in period t to be independent of the "rejected" materials and of the demands not satisfied in periods $t-2$, $t-3$. With the only exception of the service rule (described in 4.1.7.) no restrictions will be made to the conduct rules of the storage system.

4.1.7. For a demand in excess of the level of storage in a given period the store releases its whole stock. (This condition does not reduce the generality of the model since the service rule described above can be considered as a general one if the number of the sources of demands is great enough or the store enjoys monopoly).

4.2. *The definition of the parameters in the system*

The environmental parameters of the system. In the case of a particular problem the environmental parameters of the system can be considered as given by their stochastic description (probability distribution).

Let $\xi(t)$ be a discrete random variable denoting the intensity of the flow of demands emerging during this period expressed in arbitrary units of materials.

Let $p_l(t)$ denote the probability distribution of the quantity $\xi(t)$ of the demand as discrete random variable:

$$p_l(t) = P(\xi| t | = l) \qquad l = 0, 1, 2 \ldots$$

Similarly, let discrete random variable $\eta(t)$ denote the intensity of the input flow of materials in a period t and let $r_k(t)$ denote the probability distribution corresponding to $\eta(t)$:

$$r_k(t) = P(\eta| t | = k) \qquad k = 0, 1, 2 \ldots$$

To simplify notations, in the computations below the set of values of the given random variables will be considered as the set of the non-negative integers; but, out of practical considerations, a subscript N can be assumed beyond which the probabilities belonging to the corresponding values equal zero.

The technical parameters of the system. Introduction of the characteristics relating to the service system is justified by the capacity of the storage service equipment that sets a limit to the arbitrary intensities of the flow of materials.

Let $\lambda_{in}(t)$ and $\lambda_{out}(t)$ be random variables which assume non-negative integers. By introducing these values, the capacity of the input and the output service systems (that is, the maximum number of arbitrary input and output units in period t) can be modelled.

The distribution of random variables $\lambda_{in}(t)$ and $\lambda_{out}(t)$ can be considered as given by $u_n(t)$ and $v_z(t)$, thus,

$$u_n(t) = P(\lambda_{in}| t | = n) \qquad n = 0, 1, 2, \ldots$$

$$v_z(t) = P(\lambda_{out}| t | = z) \qquad z = 0, 1, 2, \ldots$$

Let positive integer C denote the capacity of the store expressed in arbitrary units of materials.* (One of the basic aspects of the described process is to be useful in decisions for the proper selection of the capacity, by providing evaluation criteria for the "goodness" of the store.)

Behaviour parameters

With the above listed system parameters already given, the following time-dependent characteristics of the system behaviour can be determined if the distribution of the initial stock is known. Let $\alpha(t)$ denote the level of stock (inventory function), that is, the quantity of materials in the store in period t, expressed in arbitrary units. Obviously, $q(t)$ is a discrete random variable if t is given. Its probability distribution is described by the following formula:

$$q_h| t | = P(\alpha| t | = h) \qquad h = 0, 1, 2, \ldots$$

Thus, probabilities $q_h(t)$ express that at the beginning of period t there is quantity h of arbitrary units in the store. Accordingly, discrete random variable $\alpha(0)$ denotes the initial stock expressed in arbitrary units. The corresponding probability distribution is:

$$q_h| 0 | = P(\alpha| 0 | = h) \qquad h = 0, 1, 2, \ldots$$

The store attempts to satisfy the demands according to the flow of materials from the supplier, the demands and the level of storage at that moment. The released quantity of materials, the intensity of the output flow of materials can also be described by discrete random variables in each given period t. Accordingly, let $\beta(t)$ denote the discrete random variable expressing the intensity of the release flow of materials, that is, the amount of output materials in period t expressed in arbitrary units. For the probability distribution of the release quantity of materials, as a discrete random variable in period t, the following denotations will be used:

$$w_s| t | = P(\beta| t | = s) \qquad s = 0, 1, 2, \ldots$$

* In the case of practical applications, the selection of arbitrary units for non-homogeneous materials constitutes the topic of a different investigation.

Table 1

Storage system parameters

Nature of system parameters	Assumed system parameters						Computed system parameters	
Classes of system parameters	Environmental parameters		Technological parameters			Behaviour parameters		
Stochastic process	Demand of customer system (expressed in arbitrary units) in period t	Quantity of arrivals (expressed in arbitrary units) in period t	Input service capacity of store (expressed in arbitrary units) in period t	Output service capacity of store (expressed in arbitrary units) in period t	(Containing) capacity of store in arbitrary units	Amount of initial stock, i.e. the quantity of materials in store (expressed in arbitrary units) in period 0	Amount of stock in store (expressed in arbitrary units) in period t	Output of store (expressed in arbitrary units) in period t
Notation	$\xi(t)$ discrete random variable	$\eta(t)$ discrete random variable	$\lambda_{in}(t)$ discrete random variable	$\lambda_{out}(t)$ discrete random variable	C positive integer	$\alpha(0)$ discrete random variable	$\alpha(t)$ discrete random variable	$\beta(t)$ discrete random variable
Distribution	$p_l(t) = P(\xi(t) = l)$ $l = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$	$r_k(t) = P(\eta(t) = k)$ $k = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$	$u_n(t) = P(\lambda_{in}(t) = n)$ $n = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$	$v_z(t) = P(\lambda_{out}(t) = z)$ $z = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$	—	$q_h(0) = P(\alpha(0) = h)$ $h = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$	$q_h(t) = P(\alpha(t) = h)$ $h = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$	$o_s(t) = P(\beta(t) = s)$ $s = 0, 1, 2, \ldots$ $t = 0, 1, 2, \ldots$

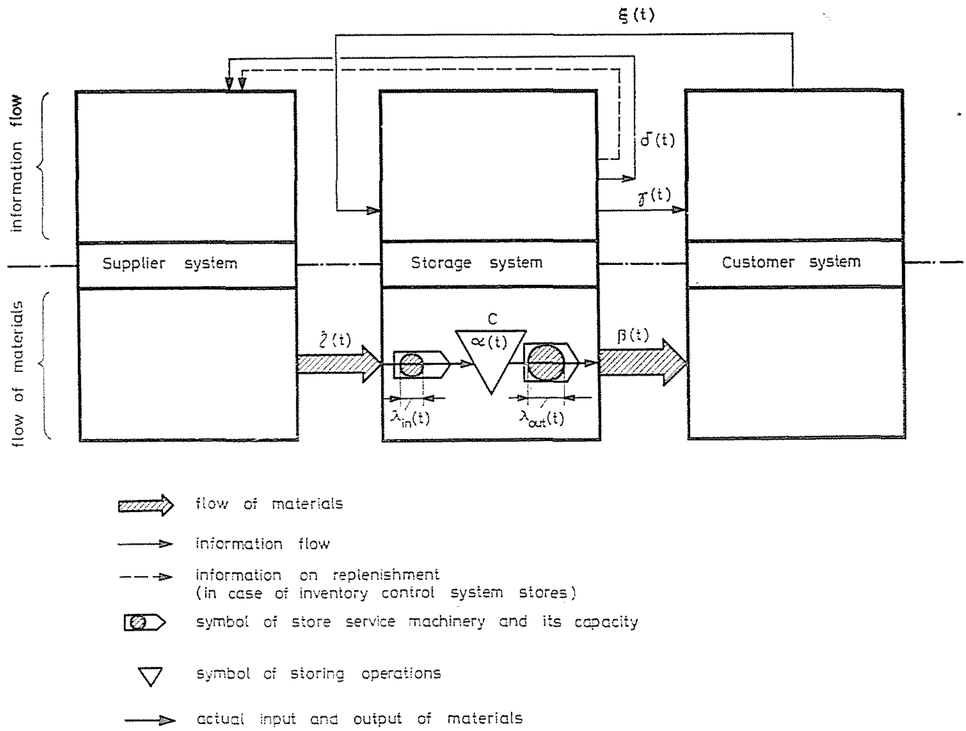

Fig. 3. System parameters and model

The system-parameters and their characteristics are compiled in Table 1.* Fig. 3 shows the systematization of the characteristics of environment, technology and flow of materials.

4.3. *Formal model for the behaviour of storage systems*

4.3.1. *Transition and output probability matrices*

Our aim is to describe the behaviour of the system with respect to a certain optimum, or evaluation conditions on the basis of the storage system parameters given in 4.2. by means of determining the following conditional probabilities:

$$p^{|t|}(h(t+1)/h(t)) \qquad (4.3.1.)$$

and

$$p^{|t|}(s(t)/h(t)) \qquad (4.3.2.)$$

* Besides these, the behaviour of the system. will be evaluated by means of the parameters $(E[\gamma(t)])$ and $(E[\delta(t)])$ denoting unmet demands, and "rejects" (5)., resp. which can be calculated from the characteristics given above.

Here 4.3.1. denotes the probability for the inventory level $h(t)$, of the store of a period t will become $h(t+1)$ at a period $(t+1)$. 4.3.2. denotes the probability that, if the inventory level is $h(t)$ the output quantity of materials will be $s(t)$ in period t. These probabilities will be explicitly defined by two matrices consisting of conditional probabilities.

Theorem 1

The storage system is assumed to satisfy all the (systems engineering) conditions in 4.1.1 to 4.1.7. In this case the inventory level of the store determines a (usually inhomogeneous) Markov chain. The elements of this transition probability matrix $\mathbf{M}(t)$ depend only on the distribution of random variables $\xi(t)$, $\eta(t)$, $\lambda_{in}(t)$ and $\lambda_{out}(t)$ and the capacity of the store, and correspond to the probability described in 4.3.1. A constituent m_{ij} of matrix $\mathbf{M}$ is:

$$m_{ij} = \sum_{l=1}^{\infty} \sum_{z=0}^{l-1} p_l|t| \cdot v_z|t| \cdot d(i,j|z) + \sum_{l=0}^{\infty} \left[p_l|t| d(i,j|l) \left(1 - \sum_{z=0}^{l-1} v_z|t| \right) \right] \quad (4.3.3.)$$

where

$$d(i,j|z) = \begin{cases} \sum_{\nu=0}^{z-i} (u_\nu + r_\nu) - \left(\sum_{\nu=0}^{z-i} r_\nu \right) \left(\sum_{\nu=0}^{z-i} u_\nu \right) \\ \qquad \text{if } j=0,\ 0 \le i \le C,\ z-i \ge 0 \\ u_{j-i+z} + r_{j-i+z} - r_{j-i+z} \cdot u_{j-i+z} - \sum_{\nu=0}^{j-i+z-1} (u_{j-i+z} \cdot r_\nu + r_{j-i+z} \cdot u_\nu) \\ \qquad \text{if } 0 < j < C,\ 0 \le i \le C,\ j-i+z \ge 0 \\ 1 - \sum_{j=0}^{C-1} \Big[u_{j-i+z} + r_{j-i+z} - r_{j-i+z} \cdot u_{j-i+z} - \\ \qquad - \sum_{\nu=0}^{j-i+z-1} (u_{j-i+z} \cdot r_\nu + r_{j-i+z} \cdot u_\nu) \Big] \\ \qquad \text{if } j=C,\ 0 \le i \le C,\ C-i+z \ge 0 \\ 0 \text{ otherwise} \end{cases} \quad (4.3.4.)$$

In the above and following formulas, for the sake of simplicity, the (period) variable "t" has been omitted and the convention adopted that, if the upper limit of a summing or an index becomes negative, the corresponding formula is zeroed.

Proof:

First, the inventory level of the store, that is, stochastic process $\alpha(t)$ is proved to be a Markov chain. To achieve this, the following has to be proved

$$P(\alpha(t) = i \mid \alpha(t-1) = j_k,$$
$$\alpha(t-2) = j_{k-1} \dots \alpha(1) = j_1) = P(\alpha(t) = i \mid \alpha(t-1) = j_k) \quad (4.3.5.)$$

The storage models can be sorted into two basic classes:

a) if the model has no "memory"* the inventory level of the store in the next period does not depend on unmet "demands" and "rejects" of previous periods,

b) in case of models with "memory" this restriction does not hold.

It is obvious that in case a), relationship (4.3.5.) holds and in case b), it simply follows from condition (4.6.1.).

In computing transition probabilities m_{ij} first let us assume that the input, output and the capacity of the store are infinite and define the probabilities $a(i, j/l)$ expressing that the inventory level will be "j" in period $t+1$ if the store had "i" amount of stock at the beginning of period t, while supposing that a quantity "l" of materials is demanded,

Applying 4.1.7:

$$a(i, j \mid l) = \begin{cases} \sum_{\nu=0}^{l-i} r_\nu & \text{if } j = 0 \text{ and } l - i \geq 0 \\ r_{j+l-i} & \text{if } j > 0 \text{ and } j + l - i \geq 0 \\ 0 \text{ otherwise} & \end{cases}$$

Namely:

- the inventory level of the stock will be zero if the amount of the input materials and the stock i equals or is short of the amount of demands,
- the inventory level of the stock will be $j > 0$ if the amount of the input material is $j + l - i = 0$
- and the probability of any other event is zero.

Now the capacity of the store input system is assumed to be n. From the aspects of the functioning of the input system, this means that any amount of materials greater than n will be rejected. Accordingly probabilities $b(i, j|l, n)$ relating to the event of the inventory level i of the store to become j, for an amount of demands l and a capacity n, are:

$$b(i, j \mid l, n) = \begin{cases} \sum_{\nu=0}^{l-i} r_\nu & \text{if } j = 0, l - i \geq 0 \text{ and } l - i < n \\ 1 & \text{if } j = 0; l - i \geq 0 \text{ and } l - i \geq n \\ r_{j+l-i} & \text{if } j > 0; j + l - i \geq 0 \text{ and } j + l - i < n \\ \sum_{\nu=n}^{\infty} r_\nu = 1 - \sum_{\nu=0}^{n-1} r_\nu & \text{if } j > 0; j + l - i \geq 0 \text{ and } j + l - i = n \\ 0 \text{ otherwise} & \end{cases}$$

* In simulating the behaviour of storage systems "memory" usually means that the store "remembers" the unmet demands and the "rejects" in the given period.

Let $d(i,j \mid l)$ denote the probability that the store transforms from level i into j, provided that the amount of demands that arrived is l and the capacity is λ_{in}.

According to the formula of total probability to this case:

$$d(i,j \mid l) = \sum_{\nu=0}^{\infty} u_\nu \cdot b(i,j \mid l,\nu).$$

Then using formula $b(i,j \mid l, n)$, summing can be described as follows:

a) if $j = 0$ and $l - i \geq 0$

$$\begin{aligned} d(i,j \mid l) &= \sum_{\nu=0}^{l-1} u_\nu + \left(\sum_{\nu=0}^{l-i} r_\nu\right)\left(\sum_{\nu=l-i+1}^{\infty} u_\nu\right) = \\ &= \sum_{\nu=0}^{l-i} (r_\nu + u_\nu) + \left(\sum_{\nu=0}^{l-i} r_\nu\right)\left(\sum_{\nu=0}^{l-i} u_\nu\right) \end{aligned} \tag{4.3.6.}$$

b) if $j > 0$ and $j + l - i \geq 0$

$$\begin{aligned} d(i,j \mid l) &= u_{j+l-i} \sum_{\nu=j+l-i}^{\infty} r_\nu + r_{j+l-i} \sum_{\nu=j+l-i+1}^{\infty} u_\nu = \\ &= u_{j+l-i}\left(1 - \sum_{\nu=0}^{j+l-i-1} r_\nu\right) + r_{j+l-i}\left(1 - \sum_{\nu=0}^{j+l-i} u_\nu\right) = \\ &= u_{j+l-i} + r_{j+l-i} - r_{j+l-i} \cdot u_{j+l-i} - \sum_{\nu=0}^{j+l-i-1} (u_{j+l-i} \cdot r_\nu + r_{j+l-i} \cdot u_\nu). \end{aligned} \tag{4.3.7}$$

c) Furthermore, if conditional inequalities 4.3.6 and 4.3.7 do not hold, the corresponding probabilities $d(i,j \mid l)$ are seen to be zero.

Let us see how probabilities $d(i,j \mid l)$ vary if the finite capacity of the store is C. In this case no changes of the inventory level where $j > C$ (that is, the corresponding $d(i,j \mid l) = 0$) are allowed. Imagining a store which, when it is full, "rejects" the arrivals the reception of which would be still allowed by its input service capacity, it is obvious that in the case of a store with a capacity C, $0 \leq i \leq C$ and $C - i + l > 0$,

$$\begin{aligned} d(i,C \mid l) &= \sum_{j=C}^{\infty} d(i,j \mid l) = \\ &= 1 - \sum_{j=0}^{C-1} \Big[u_{j+l-i} + r_{j+l-i} - r_{j+l-i} \cdot u_{j+l-i} - \\ &\quad - \sum_{\nu=0}^{j+l-i-1} (u_{j+l-i} \cdot r_\nu + r_{j+l-i} \cdot u_\nu) \Big]. \end{aligned} \tag{4.3.8}$$

For $0 \leq j < C$, relationships 4.3.6 and 4.3.7 still hold true for probabilities $d(i, j \mid l)$ even in the case of capacity C. Let now z denote the output capacity of the store in a given period. This involves the following changes for the probabilities which determine transition.

$$e(i, j \mid l, z) = \begin{cases} d(i, j \mid l) & \text{if } l < z \\ d(i, j \mid z) & \text{if } l \geq z \end{cases} \tag{4.3.9}$$

where $e(i, j \mid l, z)$ denotes the probability of the inventory level of the store to be i at the beginning and j at the end of the period, provided that the amount demanded is l and the capacity of the output service system is z. Now transition probabilities m_{ij} occurring in the theorem can be calculated. Applying the formula of total probability twice:

$$\begin{aligned} m_{ij} &= \sum_{l=0}^{\infty} p_l \left(\sum_{z=0}^{\infty} v_z \cdot e(i, j \mid l, z) \right) = \\ &= \sum_{l=0}^{\infty} \sum_{z=0}^{l-1} p_l \cdot v_z \cdot d(i, j \mid z) + \\ &+ \sum_{l=0}^{\infty} \left[p_l \cdot d(i, j \mid l) \left(1 - \sum_{z=0}^{l-1} \cdot v_z \right) \right]. \end{aligned} \tag{4.3.10}$$

Relationship 4.3.9 was applied in the transformation. $d(i, j \mid z)$ and $d(i, j \mid l)$ are determined by relationships 4.3.6, 4.3.7 and 4.3.8.

Clearly, the matrix of values m_{ij} is stochastic, since values m_{ij} constitute probability distribution for given i. This is explicit from taking into account the probabilities obtained from the process of proof and the fact that the mixing of probability distributions is also a probability distribution.

Theorem 2

The storage system is assumed to satisfy the conditions in 4.1. In this case the amounts of material released in each period can be characterized with a matrix $\mathbf{N}(t)$, and n_{ij} member of which equals probability $P^{(t)}(j \mid i)$ occurring in 4.3.2.

The elements of matrix $\mathbf{N}(t)$ are defined as:

$$n_{ij} = \sum_{l=0}^{\infty} \sum_{z=0}^{\infty} v_z \cdot p_l \cdot d(i, j \mid l, z)$$

where

$$d(i,j\,|\,l,z)=\begin{cases} u_{j-i}\cdot r_{j-i}+\sum\limits_{k=j-i+1}^{\infty}(u_{j-i}\cdot r_k+r_{j-i}\cdot u_k) & \text{if } j-i\geq 0 \text{ and } j<l,\ l\leq z \\ & \quad j-i\geq 0 \text{ and } j<z,\ z<l\leq C \\ \left(\sum\limits_{k=j-i}^{\infty} r_k\right)\left(\sum\limits_{k=j-i}^{\infty} u_k\right) & \text{if } j-i\geq 0,\ j=l,\ l\leq z\leq C \\ 1 & \text{if } j-i\leq 0,\ j=l\leq z\leq C \\ & \quad j-i\leq 0,\ j=z<l \\ \sum\limits_{v=z}^{l-1}\left[u_{v-i}\cdot r_{v-i}+\sum\limits_{k=v-i+1}^{\infty}(u_{v-i}\,r_k+r_{v-i}\,u_k)\right]+ & \\ \quad +\left(\sum\limits_{k=l-i}^{\infty} r_k\right)\left(\sum\limits_{k=l-i}^{\infty} u_k\right) & \text{if } j-i\geq 0,\ j=z<l \\ 0 \quad \text{otherwise} & \end{cases} \tag{4.3.12.}$$

Proof:

Assuming the input and output capacities to be infinite, let us describe the probabilities $a(i,j\,|\,l)$ expressing the probability that the store releases an amount i of materials if the demand is l and the level of storage is j.

From the conditions we obtain (according to service strategy) the following:

$$a(i,j\,|\,l)=\begin{cases} r_{j-i} & \text{if } j-i\geq 0 \text{ and } j<l \\ \sum\limits_{k=j-i}^{\infty} r_k & \text{if } j-i>0 \text{ and } j=l \\ 1 & \text{if } j-i\leq 0 \text{ and } j=l \\ 0 \quad \text{otherwise} & \end{cases} \tag{4.3.13.}$$

Now, let the value of the input service capacity be n. In this case we obtain probabilities $b(i,j\,|\,l,n)$ expressing that the store releases j units of materials if the inventory level is i, the demand is l and the input service capacity is n, by rewriting the formula (4.3.13.):

$$r_n \to \sum_{k=n}^{\infty} r_k \qquad \text{and}$$

$$r_{n+1}, r_{n+2}, \ldots, \to 0 .$$

After transformations, on the basis of the formula of total probability the relationship 4.3.12 is obtained, denoting probability of the event that the store releases j units of materials, provided the demand is l, the inventory level is i and the capacity is z.

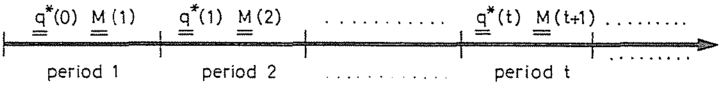

Fig. 4. Changes in inventory level distribution on discrete time scale

This results in the theorem applying twice if the formula of total probability is to values $d(i, j \mid l, z)$ to demand p_l and output capacity distributions v_z. The matrix composed of values n_{ij} is stochastic, since values n_{ij} constitute a probability distribution, for a given i.

4.3.2. *The transient behaviour of storage systems*

According to theorem 1 $\mathbf{M}(t)$ matrix, defines a *Markov chain* for random variable $\alpha(t)$ expressing the inventory level of the store and permitting to algorithmize the behaviour of the store.

Let vector $\mathbf{q}(t)$ denote the column vector of the probability distribution concerning the inventory level of the store in period t, component h of $\mathbf{q}(t)$ is:

$$q_h(t) = P(\alpha(t) = h)$$

then function of the system can be illustrated on a discrete time-scale as shown in Fig. 4. Thus the distribution of the inventory level can be determined in an arbitrary period $t + 1$, in accordance with Fig. 4, by the recursive formula:

$$\mathbf{q}^*(t + 1) = \mathbf{q}^*(t)\mathbf{M}(t + 1) \qquad (4.3.14)$$

helping to determine the probability distribution of the inventory level of the store for any period t, if distribution for the initial amount of stock $\mathbf{q}^*(o)$ is known and transition probability matrix $\mathbf{M}(t)$ is computable from theorem 1. Theorem 2. yields the probability distributions of the amount of material output in periods t.

Let vector $\mathbf{w}(t)$ denote the column vector of the probability distribution of the output of the system (the amount of materials released) in period t that is, the element s of vector $\mathbf{w}(t)$ denotes the probability:

$$w_s(t) = p(\beta(t) = s)$$

In this case the changes in the output of the system can be illustrated as in Fig. 5.

Thus, in accordance with the definition of $\mathbf{N}(t)$, the probability distribution of the amount of material released in any period t can be, defined as:

$$\mathbf{w}^*(t) = \mathbf{q}^*(t - 1)\mathbf{N}(t) \qquad (4.3.15.)$$

* (*) denotes the transposed vector (row vector) as defined by matrix algebra.

The behaviour of the system from the standpoint of the customer is shown in Fig. 6.

Summarizing our results, on the basis of theorems 1 and 2, if the environmental, technological and reliability system parameters are considered as known, transition and output probability matrices $\mathbf{M}(t)$ and $\mathbf{N}(t)$ can be stated

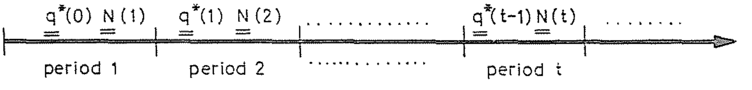

Fig. 5. Changes in output distribution on discrete time scale

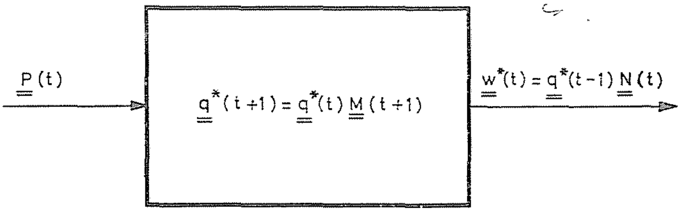

Fig. 6. Changes in storage system behaviour in terms of customer system expectations

to be constructible. The behaviour in time of the storage system, can be described by the algorithm based on relationships 4.3.14 and 4.3.15 in case of capacity C.

4.3.3. *Permanent system behaviour*

One of the basic practical features of the storage systems is the periodical change of distribution of their input parameters (distribution of demands and of input). The periodicity may be disturbed only by a stochastic "perturbation". Therefore an investigation of systems of periodical behaviour may be of particular interest. For these systems the following propositions hold:

a) Transforming the time scale by selecting one period as unit of time, matrix $\mathbf{M}(t)$ characterizing the condition of the system becomes independent from the periods. In this case the process can be described by an *ergodic Markov chain.*

The existence of ergodicity can be proved as follows: if the input distributions contain non-zero elements "in a sufficient number" then the structure of a transition probability matrix concerning a new period, wich has been obtained by multiplying the matrices $\mathbf{M}$ of the periods with each other, can be divided into four sub-matrices (See: example). Among these $\mathbf{M}_{11}$ and $\mathbf{M}_{22}$ are entirely filled with non-zero elements; $\mathbf{M}_{12}$ and $\mathbf{M}_{21}$ are triangle-matrices. The powers of these matrices are characterized by the fact that the greater the exponent, the smaller the dimension of the sub-matrices $\mathbf{M}_{12}$ and $\mathbf{M}_{21}$. Thus, a column. containing positive probabilities results only if a suitable exponent is chosen. In this case, according to the Markov theorem, the transformed chain is ergodic.

b) The behaviour of the system is characterized by the fact that, because of the choice of the probability distribution of the initial stock, transient states may occur in the beginning, but after a sufficiently long time the system is stabilized, it assumes a so-called steady state.

c) The limit distribution (absolute stationary probabilities) can be calculated by methods known from the theory of Markov chains (e.g. with the solution of simultaneous equations). Evidently distributions within the period can also be generated from the limit distribution.

5. The evaluation of the behaviour of storage systems

In modelling storage systems, a mere description of their behaviour is insufficient. Some evaluating conditions and decision rules are also necessary for the analysis of the "goodness" of their behaviour.

The given model is of help in determining the statistical characteristics of the behaviour parameters of the system yielding information necessary for the evaluation of the store.

The expected value and variance of inventory function $\alpha(t)$ in period t:

$$E[\alpha(t)] = \sum_{h=0}^{C} q_h(t) \cdot h$$

$$\sigma^2[\alpha(t)] = \sum_{h=0}^{C} [h - E(\alpha | t |)]^2 q_h(t)$$

The expected value of the amount of unmet demands $\gamma(t)$ in period t:

$$E[\gamma(t)] = E[\xi(t)] - E[\beta(t)]$$

where

$$E[\xi(t)] = \sum_{l=0}^{C} p_l(t) \cdot l \quad \text{and} \quad E[\beta(t)] = \sum_{s=0}^{C} w_s(t) \cdot s$$

The expected value of the amount of "rejected" arrivals in period t:

$$E[\delta(t)] = E[\eta(t)] + E[\alpha(t)] - E[\alpha(t+1)] - E[\beta(t)]$$

Table 2

Parameters for the evaluations of storage system behaviour

Stochastic process	Notation	Statistical parameters
Inventory function	$\alpha(t)$	$E[\alpha(t)] = \mathbf{q}^*(t-1)\mathbf{M}(t)\mathbf{k}$
		$\sigma^2[\alpha(t)] = \sum_{h=0}^{C} [h - E(\alpha(t))]^2 q_h(t)$
Rate of unmet demands	$\gamma(t)$	$E[\gamma(t)] = [(\mathbf{p}^* - \mathbf{q}^*(t-1)\mathbf{N}(t)\mathbf{M}(t)]\mathbf{k}$
"Rejects" (arrivals of which entry into the storage system is denied)	$\delta(t)$	$E[\delta(t)] = [r^* + q^*(t-1)[\mathbf{I} - \mathbf{M}(t+1) - \mathbf{N}(t)] \cdot \mathbf{M}(t)]\mathbf{k}$

where

$$E[\delta(t)] = \sum_{k=0}^{\infty} r_k(t) \cdot k \quad \text{and} \quad \alpha(t+1) = \alpha(t) - (\eta(t) - \delta(t)) - \beta(t)$$

These relationships can also be formulated by matrix formalism.

If $\mathbf{k}^*$ denotes vector $(0, 1, 2, \ldots C)$ then

$$E[\alpha(t)] = \mathbf{q}^*(t-1)\mathbf{M}(t)\mathbf{k}$$
$$E[\gamma(t)] = (\mathbf{p}^* - \mathbf{q}^*(t-1)\mathbf{N}(t)\mathbf{M}(t))\mathbf{k}$$
$$E[\delta(t)] = [\mathbf{r}^* + \mathbf{q}^*(t-1)(\mathbf{I} - \mathbf{M}(t+1) - \mathbf{N}(t))\mathbf{M}(t)]\mathbf{k}$$

(here **p** and **r** denote the vectors of the probability distributions corresponding to the amount of demands and of arrivals, respectively; **I** is a unit matrix with dimension C).

The "goodness" of the system can be characterized on the basis of the obtained results, by evaluation conditions and decision rules. Undoubtedly, no universal rules can be given for evaluating the "goodness" of the store. During the process of designing or investigating the store, suitable evaluating conditions (optima) have to be selected with respect to the peculiarities of the actual situation as well. In the investigation of the "goodness" of the system, if it is an input-oriented "producing" store the expected values $E(\alpha|\, t\, |)$, $E(\eta|\, t\, |)$ characterizing the quality of the input should be considered while, if it is a demand — oriented "trading" store the expected value $E[\beta(t)]$ that concerns rapid and precise releases is to be investigated.

As an example the following functional reliability values can be ment'oned similar to the notion of "service grade" in the theory of inventory control.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \frac{E[\beta(t)]}{E[\xi(t)]} \leq 1-\varepsilon \tag{5.1.}$$

$$\frac{E[\beta(t)]}{E[\xi(t)]} \leq 1-\varepsilon. \tag{5.2.}$$

Formula 5.1. shows that in case of an interval consisting of an arbitrary number of time units the rate of the probable value of average satisfied demands in one period t is in each period t higher than, or equal to an arbitrary $1 - \varepsilon$ value. Formula 5.2. shows in one interval T the rate of the probable value of average satisfied demands in each period to be higher than, or equal to an optional $1 - \varepsilon$ value. The characteristics suitable for the evaluation of the storage system are compiled in Table 2.

6. Example

For illustrating the described model let us present a concrete arithmetic problem emerged during the computer research on the system model. We have observed the behaviour of the system during 50 periods with the following parameters:

1. capacity C of the store is 11 arbitrary units of materials;
2. the initial stock comprises 6 arbitrary units of materials at a probability 1;
3. the probability distribution of the input service system has been assumed to have a service system of 3 machines each of a capacity of 2 units of materials/period. The probability has been taken into consideration that the machines could go wrong; according to [1] it is $\mathbf{v}$ = [0.0008, 0.0068, 0.0834, 0.2245, 0.1578, 0.000, 0.5259, 0.000, 0.000, 0.000, 0.000]
4. distribution of the capacity of the output service system can be obtained similarly to that of the input service system, but we have assumed to have 2 handling machines instead of 3. $\mathbf{u}$ = [0.0083, 0.0496, 0.3636, 0.000, 05785, 0.000, . . .]
5. demands arrive in every period according to the Poisson distribution of 2 expected values.
6. the Poisson distribution of the amount of arrivals has 6 expected values in every 3 periods, it is 0 in other periods with 1 probability.

Tables 3 and 4 show the transition probability matrices and the output probability matrices of the system.

Table 3

Transition probability matrix $\mathbf{M}(t)$ in periods 0,3,6 . . .

	0	1	2	3	4	5	6	7	8	9	10
0	0.1821	0.1679	0.2041	0.1581	0.1562	0.0940	0.0376	0.0000	0.0000	0.0000	0.0000
1	0.0815	0.1006	0.1679	0.2041	0.1581	0.1562	0.0940	0.0376	0.0000	0.0000	0.0000
2	0.0300	0.0515	0.1006	0.1679	0.2041	0.1581	0.1562	0.0940	0.0376	0.0000	0.0000
3	0.0071	0.0229	0.0515	0.1006	0.1679	0.2041	0.1581	0.1502	0.0940	0.376	0.0000
4	0.0023	0.0048	0.0229	0.0515	0.1006	0.1679	0.2041	0.1581	0.1562	0.0940	0.0376
5	0.0000	0.0023	0.0048	0.0229	0.0515	0.1006	0.1679	0.2041	0.1581	0.1562	0.1316
6	0.0000	0.0000	0.0023	0.0048	0.0229	0.0515	0.1006	0.1679	0.2041	0.1581	0.2877
7	0.0000	0.0000	0.0000	0.0023	0.0048	0.0229	0.0515	0.1006	0.1679	0.2041	0.4458
8	0.0000	0.0000	0.0000	0.0000	0.0023	0.0048	0.0229	0.0515	0.1006	0.1679	0.6500
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0023	0.0048	0.0229	0.0515	0.1006	0.8179
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0023	0.0048	0.0229	0.0515	0.9185

"Output probability" matrix $\mathbf{N}(t)$ in periods 0,3,6 . . .

	0	1	2	3	4	5	6	7	8	9	10
0	0.1616	0.3034	0.3790	0.1093	0.0467	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.1373	0.3156	0.3683	0.1099	0.0689	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.1373	0.2997	0.3802	0.1039	0.0790	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.1373	0.2997	0.3748	0.1074	0.0808	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4

Transition probability matrix M(t) in periods 1,2,4,5 . . .

	0	1	2	3	4	5	6	7	8	9	10
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.8627	0.1373	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.5630	0.2997	0.1373	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.1882	0.3748	0.2997	0.1373	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.0831	0.1050	0.3748	0.2997	0.1373	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0000	0.0831	0.1050	0.3748	0.2997	0.1373	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0000	0.0000	0.0831	0.1050	0.3748	0.2997	0.1373	0.0000	0.0000	0.0000	0.0000
7	0.0000	0.0000	0.0000	0.0831	0.1050	0.3748	0.2997	0.1373	0.0000	0.0000	0.0000
8	0.0000	0.0000	0.0000	0.0000	0.0831	0.1050	0.3748	0.2997	0.1373	0.0000	0.0000
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0831	0.1050	0.3748	0.2997	0.1373	0.0000
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0831	0.2997	0.1050	0.3748	0.1373

"Output probability" matrix N(t) in periods 1,2,4,5 . . .

	0	1	2	3	4	5	6	7	8	9	10
0	0.1616	0.3034	0.3790	0.1093	0.0467	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.1373	0.3156	0.3683	0.1099	0.0689	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.1373	0.2997	0.3802	0.1039	0.0790	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.1373	0.2997	0.3748	0.1074	0.0808	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1373	0.2997	0.3748	0.1050	0.0831	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

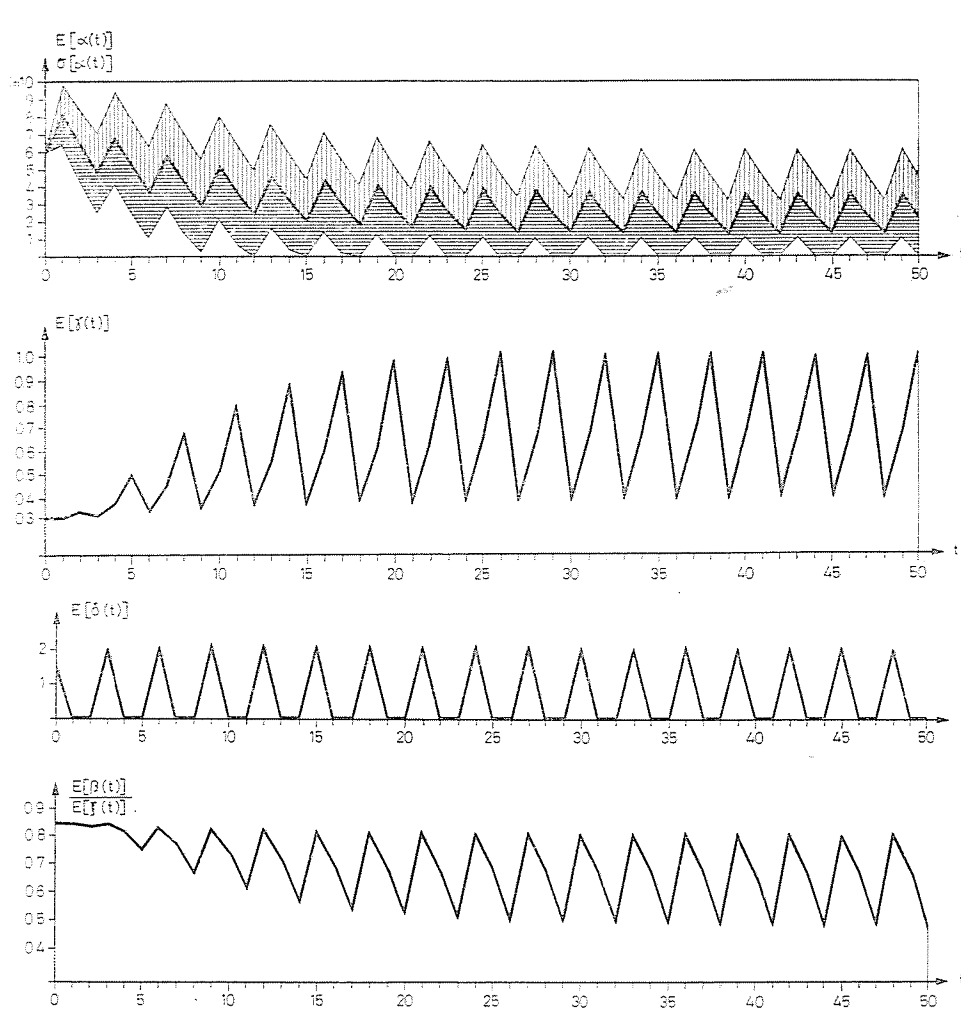

Fig. 7. Diagram showing the changes in storage system behaviour parameters (Example 1)

Fig. 7 illustrates the changes in the parameter values characterizing the behaviour of the system, such as the expected value and variance of the inventory level, the expected value of the amount of unmet demands, the expected value of "reject" and the degree of functional reliability as related to time. (Variance is marked by shading in the figure). If the capacity of the service system is characterized by the distributions [4 and 1 machines]

$\mathbf{v} = [0.0001, 0.0008, 0.0142, 0.0820, 0.2336, 0.000, 0.1912, 0.000, 0.4781, 0.000, 0.000]$

and $\mathbf{u} = [0.0909, 0.2727, 0.6364, 0.000, \ldots, 0.000]$ and characteristics 1, 2, 5, 6 remain the same, the changes in the parameters of the storage system are shown in Fig. 8.

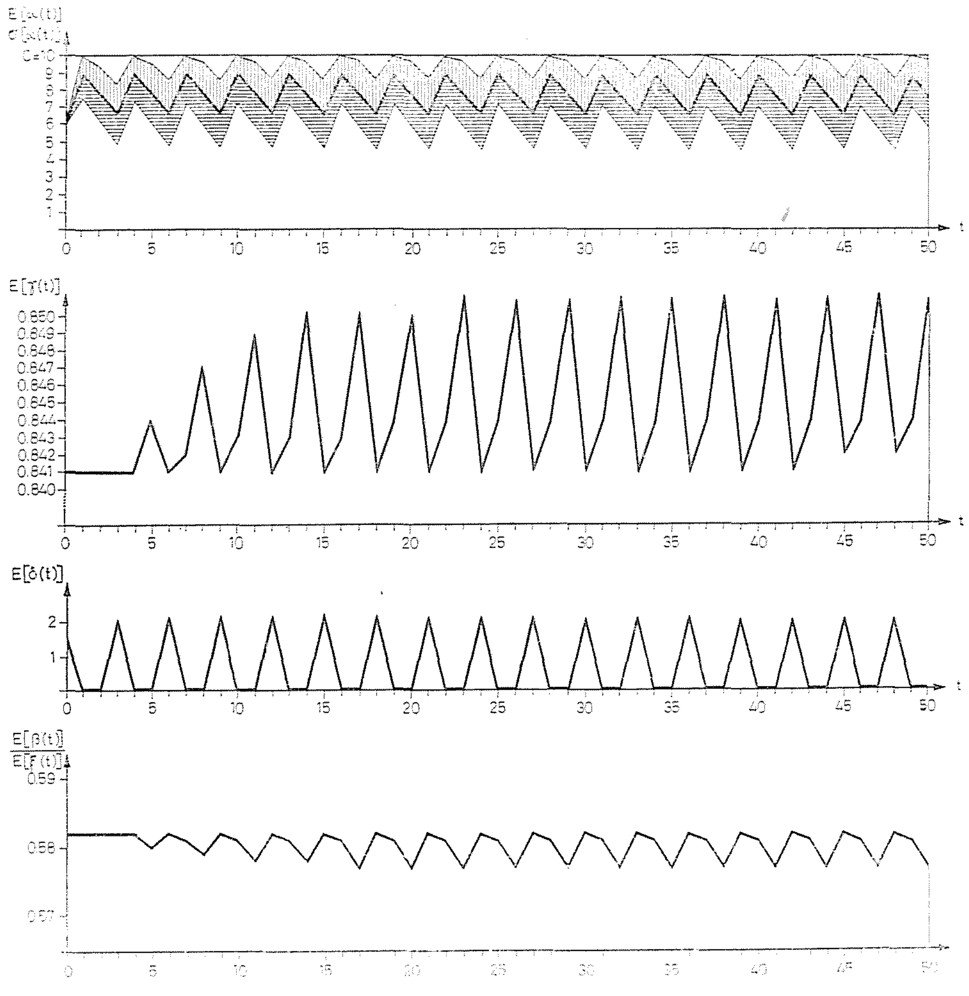

Fig. 8. Diagram showing the changes in storage system behaviour parameters (Example 2)

7. Application and further perspectives

The basic possibilities for applying the described system model and algorithm, and, also, the simulation research based on them can be summarized as follows:

1. The investigation of the system can play a prominent role in the preliminary phases of store developement decisions. In this case the effect of the proposed changes (e.g. the increase in the capacity and reliability of the system) and the behaviour of the storage system can be calculated (and evaluated) from theoretical relationships described above.

2. In the course of designing large scale systems beside the expected environmental influences (stochastic characteristics of arrivals and flow of demands) technological and reliability system parameters required to achieve a certain degree of "goodness" (capacity of the store, capacity and reliability of service systems) can also be defined. The obtained parameters (parameter variants) will mean the basic information for determining the capacity of the store, the selection of the service systems and establishing the material flow processes.

The great amount of computations involved in the course of investigations and design justifies computerization.

The following cases of possible application provide opportunities for further research and design.

1. The capacity of storage service systems as a stochastic process is mostly influenced by the changes in time of the reliability parameters of the service systems. The included stochastic process permits to apply the model for investigating the effect of service-system amortization on the "goodness" of the store.
2. In designing the store the model given here provides an opportunity to take into account the trends to increase the flow of demands and arrivals, thus enabling us to make up for the changes maintaining the capacity level needed, at the desired degree of the "goodness" of the store ("elastic model").

The possibilities for further investigations of storage system behaviour can be summarized on the basis of the system model as follows:

1. In case of more than one kind of products the system model can be generalized by usual mathematical procedures.
2. From the aspect of simulating the system behaviour, it seems often justified to assume that within the storage service system the output system is separate from the input system (e.g. for what are called *Komissionierlagern* in German terminology). Storage service systems can however not often be divided into input and output service systems, that is, the input and the output capacity levels of the system cannot be separated.
   In this case a unified service system operates according to a given strategy. This can be simulated by determining a random variable that controls the changes of the input and output intensity values in each period.
   The latter model can be deduced from the system model of behaviour, but to obtain more detailed results further investigations are necessary.
3. If the flow of demands and the input flow of materials are independent of time or they are periodical, the inventory function and the output function of the store can be considered as the state function and output function

of a *stochastic automaton*. The stochastic automaton yields a simple algorithm for the definition of both the probability distribution of the inventory level and the output at any point of time.

4. The general system model provides the opportunity to construct models with memory. In this case the store "remembers" the information concerning unmet demands, and tries to satisfy the postponed demands in subsequent periods. Here the description of the accumulating amount of unmet (postponed) demands in terms of the technological and reliability parameters means to investigate the system behaviour.
5. The investigation of inventory control storage systems with memory seems to be a complex research problem of incorporating inventory control strategies in the behaviour model with memory. In this case investigations will focus on the closed circuit of demans and materials.*
6. By making period Δt to tend to 0 in the general model a possibility is given to characterize the storage system by a non-discrete model. This mathematical generalization may yield useful "rheological" rules for the behaviour of storage systems.
7. Finally, the system model provides the opportunity for an extensive development in the investigations into storage system behaviour and in the description of its self-control processes by means of adaptive and learning algorithms.

Summary

A stochastic model for the description and evaluation of storage system behaviour. By uniting inventory control and dam model approaches with the description of internal store processes the described stochastic model aims at considering all the imporant technological factors involved and at describing storage system behaviour in relation to time and in its interrelation with its environment. The model and the algorithm given here are of use in designing stores and in investigations, developments and investment decisions concerning particular storage systems.

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* What sort of target function can be constructed of the parameters characterizing the behaviour of the system and calculated by the given model, and how the corresponding problem of the optimum can be solved — are problems to be investigated subsequently.