CONVERGENCY OF THE METHOD OF SUCCESSIVE APPROXIMATIONS APPLIED TO AN INHOMOGENEOUS ALGEBRAIC SYSTEM OF EQUATIONS

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1. Introduction

In technical practice often solutions for inhomogeneous linear algebraic systems of equations are needed.

Among others, this is the case where a differential equation is to be solved by a numerical procedure based on discretization, or where solutions of homogeneous or inhomogeneous linear differential equations and systems of differential equations are to be fitted to initial conditions.

Direct methods are affected by an inherent inaccuracy that proved to be reducible by supplementary iteration, requiring a computer of an arithmetic with a double word-length. [4]

When successive approximation is applied, the question of convergency for an arbitrary initial unknown vector emerges. Below some remarks are made on the convergency condition of the method of successive approximations.

2. Successive approximations and convergency

The inhomogeneous linear algebraic system of equations

$$\mathbf{A}x = b$$

where $\mathbf{A}$ and b are known, x is the unknown vector, which is wanted. n is the number of the equations. Let us assume that

1. $\mathbf{A}$ is quadratic
2. $n < \infty$
3. $\exists \; a_{ij} \in \mathbf{A} : a_{ij} \neq 0 \qquad i, j = 1, 2 \ldots n$
4. $\forall \; a_{ij} \in \mathbf{A} : a_{ij} < \infty \qquad i, j = 1, 2 \ldots n$

The method of successive approximations determines the $(k+1)$-th iterate from the k-th ([4]):

$$x^{(k+1)} = (\mathbf{E} - \mathbf{A})x^{(k)} + b$$

where $\mathbf{E} = \langle 1, 1 \dots 1 \rangle$ is the identity matrix.

The successive approximations converge at an arbitrary initial vector if ([4]):

$$\| \mathbf{E} - \mathbf{A} \| < 1 .$$

From the definition of spherical norm

$$\| \mathbf{E} + \mathbf{A} \| = \sqrt{\sum_{i=1}^{n} (1 - a_{ii})^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^2} .$$

For a given matrix $\mathbf{A}$ this condition is not always satisfied, but in case of a coefficient matrix with certain properties the system of equations can be transformed to satisfy the criterion of convergency.

The first term under the root-sign written in detail:

$$\sum_{i=1}^{n} (1 - a_{ii})^2 = \sum_{i=1}^{n} (1 - 2a_{ii} + a_{ii}^2) .$$

As this sum is finite

$$\sum_{i=1}^{n} (1 - 2a_{ii} + a_{ii}^2) = n - 2 \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} a_{ii}^2$$

and

$$\| \mathbf{E} - \mathbf{A} \| = \sqrt{n - 2 \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} \sum_{j=1}^{n} a^2_{ij}} .$$

For the sake of simplicity let us introduce the notations

$$\beta = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad \vartheta = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 .$$

In order to get a root smaller than 1, the amount under the root-sign must be smaller than 1

$$1 > n - 2\beta + \vartheta .$$

If both sides of the system of equations are identically modified only the last two terms will change. A factor is sought for the system of equations to give a product satisfying the criterion of convergency, or it has to be proven, that if the coefficient matrix does not satisfy the condition defined below, no such factor exists. Of course, equations satisfying a priori the criterion of the convergency of successive approximations are not subject of discussion.

It is known that the right and left side of an equation can be multiplied with the same number except zero. Let c be this non-zero multiplie, hence:

$$1 - n > - 2c\beta + c^2\vartheta .$$

For a given system of equations that c is wanted for which the right side of the inequality above has a minimum. Minimum exists at a c where:

$$\frac{d}{dc}(c^2\vartheta - 2c\beta) = 0$$

and it is minimum indeed, if

$$\frac{d^2}{dc^2}(c^2\vartheta - 2c\beta) > 0 .$$

The second condition is obviously satisfied

$$2\vartheta = 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 > 0 .$$

The first condition yields the equation

$$2c\vartheta - 2\beta = 0$$

from which

$$c = \frac{\beta}{\vartheta} .$$

Substituting this into the expanded form of the condition of convergency we get

$$1 - n > \left(\frac{\beta}{\vartheta}\right)^2 \vartheta - 2\frac{\beta}{\vartheta}\beta .$$

Multiplying this by (—1), after possible simplifications:

$$n - 1 < \frac{\beta^2}{\vartheta} = \alpha .$$

If this condition is satisfied, there is at least one number c, multiplying by which both sides of the equation the applied successive approximation will be convergent for an arbitrary initial vector. The resulting equation is

$$cAx = cb .$$

3. Flow chart for a computer

A computer program was prepared for testing the procedure above in practice; the program is presented in the Appendix. The program has been written in ALGOL 1204 $\subset$ ALGOL 60 language ([3]), for an ODRA 1204 type computer. ([1])

The flow chart below contains the principal steps of the program:

Flow chart:

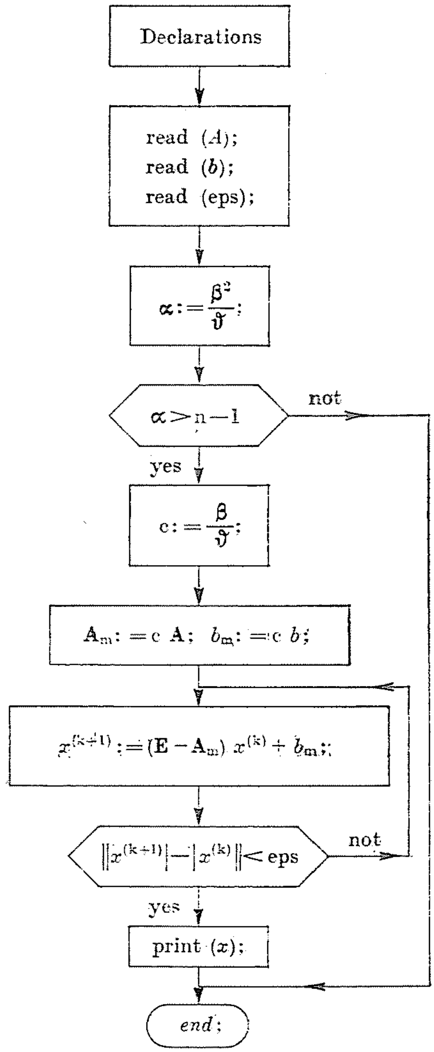

Fig. 1

4. Numerical example

Let the equation

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 5 \\ 7 \\ 1 \end{vmatrix}$$

be solved by successive approximations making use of the computer program. Obviously

$$\|\mathbf{E} - \mathbf{A}\| = \sqrt{11} > 1$$

thus the program cannot be applied to equation in the present form. But as

$$\alpha = \frac{\left(\sum_{i=1}^{n} a_{ii}\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} \cong 2.2273$$

and $n = 3$, hence $\alpha > n - 1$ and the equation can be transformed into $\|\mathbf{E} - c\mathbf{A}\| < 1$. The multiplie is:

$$c = \frac{\sum_{i=1}^{n} a_{ii}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} \cong 0.31818$$

and the procedure applied to the equation $c\mathbf{A}x = c \cdot b$ is convergent at an arbitrary initial vector since

$$\|\mathbf{E} - c\mathbf{A}\| \cong 0{,}933 < 1 .$$

Of course, the program provides the solution of the equations:

$$x = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix} .$$

Summary

A simple criterion was needed to decide whether a system of equations is convergent or, can be transformed into a convergent one for successive approximations.

With the coefficient matrix at hand the criterion $\alpha > n - 1$ easily and quickly determines convergency.

APPENDIX

```
'COMMENT AERO ES TERMOTECHNIKA TANSZEK ;
'BEGIN 'INTEGER I,J,R,N;
READ (N);
'BEGIN 'REAL D1,D2.D3,EPS,C:
        'ARRAY X,XX,Y,B[1:N],
                A[1:N,1:N];
READ (EPS); READ (B); READ (A);
D1:=D2:=0:
```

```
'FOR I:=1 'STEP 1 'UNTIL N 'DO 'BEGIN
  D1:=D1+A[I,I];
    'FOR J:=1 'STEP 1 'UNTIL N 'DO
    D2:=D2+A[I,J] ↑ 2
  'END;
D3:=D1*D1/D2;
FORMAT ("!!11.111⏨+1$);
PRINT ("
      A KONVERGENCIA FELTETELE:
  $,D3," NAGYOBBMINT $,N-1,"
    $);
'IF D3<N-1 'THEN 'GOTO VEGE;
C:=D1/D2;
PRINT ("
      A KONVERGENCIAT BIZTOSITO
      TENYEZO: $,C);
LINE (3);
'FOR I:=1 'STEP 1 'UNTIL N 'DO 'BEGIN
  'FOR J:=1 'STEP 1 'UNTIL N 'DO A[I,J]:=C*A[I,J];
B[I]:=C*B[I]: x[i]: =0.1  'END;
'FOR R:=0,R 'WHILE R<1 'DO 'BEGIN D1:=D2:=0;
  'FOR I:=1 'STEP 1 'UNTIL N 'DO 'BEGIN Y[I]:=0;
    'FOR J:=1 'STEP 1 'UNTIL N 'DO
    Y[I]:=Y[I]+A[I,J]*X[J];
    XX[I]:=X[I]-Y[I]+B[I];
    D1:=D1+X[I] ↑ 2;
    D2:=D2+XX[I] ↑ 2 'END;
  'IF ABS ( SQRT (D2)-SQRT (D1))<EPS 'THEN R:=R+2
'ELSE 'BEGIN 'FOR I:=1 'STEP 1 'UNTIL N 'DO
                X[I]:=XX[I]
'END 'END;
PRINT ("
    VEGEREDMENY
    $);
FORMAT ("?!!+1.11111⏨+1$);
'FOR I:=1 'STEP 1 'UNTIL N 'DO PRINT (XX[I]);
LINE (3);
VEGE:LINE (1) 'END 'END; ?
```

References

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4. Rózsa, P.: Linear Algebra and Its Applications.* Műszaki Könyvkiadó, Budapest, 1974

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