DETERMINATION OF THE EQUATION OF THE BALLOON CURVE FORMED BY THE UNWINDING YARN IN THE SHUTTLE DURING WEAVING

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Received June 3, 1972

Presented by Prof. Dr. G. SZÁSZ

In weaving, a balloon is formed by the yarn unwinding from the pirn in the shuttle. At first the balloon is of moderate height, it has, however, a tendency to increase. With full pirn the maximum diameter of the balloon is small but with unwinding its length may attain that of the inner diameter of the shuttle wall.

Since yarn tension and the number of weft breakages, thus loom efficiency are affected by changes in the form of the balloon curve, it is of interest to know exactly these changes.

Let us derive the equation of the balloon curved formed by the unwinding yarn in the shuttle.

The forces acting on the small arch element ds of the yarn balloon are as follows:

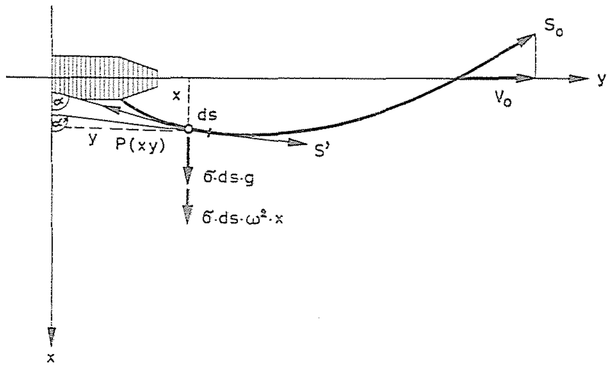
a) centrifugal force,
b) force of gravity,
c) tensions stretching the arch element ds,
d) air resistance,
e) friction on the shuttle wall.

Be σ the mass of unit length of the yarn; ds the length of the arch element; g the gravity acceleration; ω the average angular velocity of rotation of the arch element; S and S' the stretching forces acting at the extremities of the arch element; α and α' the angles between the stretching forces and the co-ordinate axis x.

Since the mass of the yarn element ds is $\sigma \cdot ds$, the force of gravity acting on it is $\sigma \cdot ds \cdot g$. Be x the distance between the arch element ds and the rotation axis, then the centrifugal force acting on the arch element amounts to $\sigma \cdot ds \cdot \omega^2 x$, furthermore two stretching forces of opposite sense, S and $S' = S + dS$ are also acting on the arch element, causing the yarn to be steadily stretched. Finally, the air resistance acting on the arch element is proportional to the length of the arch element and — because of the high velocity — to the n^{th} power of the velocity, thus $c(x\omega)^n \cdot ds$, where c is a factor proportionate to the air resistance.

In order to determine the equation of the yarn curve let us assume that neither friction nor air resistance occur, thus for the purpose of calculation the yarn curve can be considered as a plane curve.

Be the y axis of the co-ordinate system along the axis of the pirn and take the arch element ds of the yarn at the point $P(xy)$.



The arch element can only be balanced if the algebraic sum of the forces acting on it is zero, thus for the components of the forces along the axes x and y the respective conditions

$$\Sigma X = 0$$

$$\Sigma Y = 0$$

are to be satisfied.

Rendering positive the component of the force S along the axis x, the force will be of opposite sense

$$\sigma.\ ds.\ \omega^2.\ x + \sigma.\ ds.\ g \quad - S \cos \alpha + S' \cos \alpha' = 0$$

$$- S \sin \alpha + S' \sin \alpha' = 0,$$

where α and α' are angles betwen the forces S and S' and the positive direction of the axis x. Expanding in series the functions $S' \cos \alpha'$, and $S' \sin \alpha'$:

$$S' \cos \alpha' = S \cos \alpha + \frac{d}{dx} (S \cdot \cos \alpha) dx + \ldots$$

$$S' \sin \alpha' = S \sin \alpha + \frac{d}{dx} (S \cdot \sin \alpha) dx + \ldots$$

respectively.

Thus, the equilibrium equations can be written as:

$$\sigma . ds . \omega^2 . x + \sigma \cdot ds \cdot g + \frac{d}{dx}(S \cdot \cos \alpha) dx = 0 \text{ and}$$

$$\frac{d}{dx}(S \cdot \sin \alpha) dx = 0.$$

Since the stretching forces S and S' are of the direction of the tangent to the curve and the tangent of the tangent direction is $\operatorname{tg} \alpha = y'$,

$$\cos \alpha = \frac{1}{\sqrt{1 + \operatorname{tg}^2 \alpha}} = \frac{1}{\sqrt{1 + y'^2}}$$

$$\sin \alpha = \frac{1}{\sqrt{1 + \operatorname{ctg}^2 \alpha}} = \frac{\operatorname{tg}}{\sqrt{1 + \operatorname{tg}^2 \alpha}} = \frac{y'}{\sqrt{1 + y'^2}}$$

the equilibrium equations take the following form:

$$\sigma . ds . \omega^2 . x + \sigma . ds . g + \frac{d}{dx}\left[S \cdot \frac{1}{\sqrt{1 + y'^2}} \right] dx = 0$$

$$\frac{d}{dx}\left[S \cdot \frac{y'}{\sqrt{1 + y'^2}} \right] dx = 0 .$$

Integrating the second equation along x:

$$\frac{S \cdot y'}{\sqrt{1 + y'^2}} = C_1.$$

And from the first equation:

$$\sigma \cdot \frac{ds}{dx}(\omega^2 x + g) + \frac{d}{dx}\left[\frac{S}{\sqrt{1 + y'^2}}\right] = 0 .$$

Since

$$\frac{ds}{dx} = \sqrt{1 + y'^2},$$

$$\sigma \cdot \sqrt{1 + y'^2}(\omega^2 x + g) + \frac{d}{dx}\left[\frac{S}{\sqrt{1 + y'^2}}\right] = 0 .$$

Integrating the latter equation along x:

$$c_2 - \sigma \int (\omega^2 x + g) \sqrt{1 + y'^2}\, dx = \frac{S}{\sqrt{1 + y'^2}} .$$

From the former equation, however,

$$\frac{S}{\sqrt{1 + y'^2}} = \frac{c_1}{y'} ,$$

thus

$$c_2 - \sigma \int (\omega^2 x + g) \sqrt{1 + y'^2}\, dx = \frac{c_1}{y'} .$$

The latter equation is the differential equation of the balloon curve formed by the yarn in the shuttle.

Integrating the equation along x:

$$- \sigma (\omega^2 x + g) \sqrt{1 + y'^2} = - \frac{c_1}{y'^2} \cdot y''$$

a second order non-linear but incomplete differential equation.

With the transformations $y' = p$ and $y'' = \frac{dp}{dx}$

$$\sigma(\omega^2 x + g)\sqrt{1 + p^2} = \frac{c_1}{p^2} \cdot \frac{dp}{dx},$$

or

$$\int \frac{dp}{p^2 \sqrt{1 + p^2}} = \frac{\sigma}{c_1} \cdot \int (\omega^2 x + g) dx.$$

Substituting the left side by $p = shu$

$$\int \frac{chu\, du}{sh^2 u \sqrt{1 + sh^2 u}} = \frac{\sigma}{c_1} \cdot \int (\omega^2 x + g) dx,$$

or

$$\int \frac{du}{sh^2\, u} = \frac{\sigma}{c_1} \cdot \int (\omega^2 x + g) dx.$$

After integration we may write:

$$- cthu = - \frac{\sqrt{1 + sh^2 u}}{shu} = \frac{\sigma}{c_1} \left(\frac{\omega^2}{2}\, x^2 + gx \right) + c_3 .$$

Raising both sides to the second power:

$$\frac{1}{y'^2} + 1 = \left[\frac{\sigma}{c_1} \left(\frac{\omega^2}{2}\, x^2 + gx \right) + c_3 \right]^2 ,$$

whence

$$y' = \frac{dy}{dx} = \frac{1}{\sqrt{\left[\frac{\sigma}{c_1} \left(\frac{\omega^2}{2}\, x^2 + gx \right) + c_3 \right]^2 - 1}} .$$

The equation of the balloon plane curve is of the form:

$$y = \int \frac{dx}{\sqrt{\left[\frac{\sigma}{c_1} \left(\frac{\omega^2}{2} x^2 + gx \right) + c_3 \right]^2 - 1}} + K .$$

The function under the integral below the root contains the fourth degree expression of the variable x, thus the integral is elliptical. In order to transform it to canonical form let us determine first the values of the integration constants c_1, c_3 and K.

Since in the apex of the balloon $x = 0$,

$$-\frac{\sqrt{1+y'^2}}{y'} = c_3.$$

However, $y' = \operatorname{tg} \alpha$ represents the direction tangent of the tangent at the point $A(0, H)$ and along the peak tension S_0, thus

$$-\frac{\sqrt{1+tg^2 \alpha}}{\operatorname{tg} \alpha} = -\frac{\frac{1}{\cos \alpha}}{\frac{\sin \alpha}{\cos \alpha}} = \frac{1}{\sin \alpha} = -\frac{1}{\sin (90^\circ + \gamma_0)} =$$

$$= -\frac{1}{\cos \gamma_0} = -\frac{1}{\frac{V_0}{S_0}} = -\frac{S_0}{V_0} = c_3 .$$

On the other hand, referred to point $A(0, H)$ from the equation

$$\frac{S \cdot y'}{\sqrt{1+y'^2}} = c_1$$

we may write

$$\frac{S_0 \cdot (y')_A}{\sqrt{1+(y'^2)_A}} = S_0 \cdot \frac{\frac{\sin \alpha}{\cos \alpha}}{\frac{1}{\cos \alpha}} = S_0 \sin \alpha =$$

$$= S_0 \sin (90^\circ + \gamma_0) = S_0 \cos \gamma_0 = S_0 \cdot \frac{V_0}{S_0} = V_0 = c_1.$$

Finally, let us calculate the integration constant K from the equation of the balloon curve. According to the equation of the balloon curve, at point $A(0, H)$

$$H = \int_a^0 \frac{dx}{\sqrt{\left[\frac{\sigma}{c_1}\left(\frac{\omega^2}{2} x^2 + gx\right) + c_3\right]^2 - 1}} + K ,$$

furthermore

$$y = \int_{a}^{x} \frac{dx}{\sqrt{\left[\frac{\sigma}{c_1}\left(\frac{\omega^2}{2} x^2 + gx\right) + c_3\right] - 1}} + K.$$

From the difference of the equations

$$y - H = \int_{0}^{x} \frac{dx}{\sqrt{\left[\frac{\sigma}{V_0}\left(\frac{\omega^2}{2} x^2 + gx\right) - \frac{S_0}{V_0}\right]^2 - 1}},$$

or

$$y = \int_{0}^{x} \frac{dx}{\sqrt{\left[\frac{\sigma}{V_0}\left(\frac{\omega^2}{2} x^2 + gx\right) - \frac{S_0}{V_0}\right]^2 - 1}} + H.$$

The transformation of the elliptical integral to canonical form requires the determination of the roots of the fourth-degree polynomial contained in the denominator. Resolving the polynomial into factors we get

$$\left[\frac{\sigma}{V_0}\left(\frac{\omega^2}{2} x^2 + gx\right) - \frac{S_0}{V_0} + 1\right] \cdot \left[\frac{\sigma}{V_0}\left(\frac{\omega^2}{2} x^2 + gx\right) - \frac{S_0}{V_0} - 1\right] = 0.$$

Summarizing the roots of both factors we may write:

$$x = -\frac{g}{\omega^2} \pm \sqrt{\left(\frac{g}{\omega^2}\right)^2 + 2 \cdot \frac{S_0 \pm V_0}{\sigma\,\omega^2}},$$

where from the double sign under the root the upper one relates to the first factor of the fourth degree polynomial, while the lower one to its second factor.

V_0 being the component of the peak tension S_0 along the axis y, there is always $V_0 = S_0$, thus the expression under the root is always positive, and consequently the fourth-order polynomial considered has four real roots. Writing the roots in detail

$$x_{1,2} = -\frac{g}{\omega^2} \pm \sqrt{\left(\frac{g}{\omega^2}\right) + 2 \cdot \frac{S_0 - V_0}{\sigma\,\omega^2}}$$

$$x_{3,4} = -\frac{g}{\omega^2} \pm \sqrt{\left(\frac{g}{\omega^2}\right)^2 + 2 \cdot \frac{S_0 + V_0}{\sigma\,\omega^2}}.$$

Let us determine the order of magnitude of the roots. Be the highest root denoted by a_4, and the lowest one by a_1. For the sake of shortness introducing the notations

$$A = \sqrt{\left(\frac{g}{\omega^2}\right)^2 + 2 \cdot \frac{S_0 - V_0}{\sigma \omega^2}},$$

$$B = \sqrt{\left(\frac{g}{\omega^2}\right)^2 + 2 \cdot \frac{S_0 + V_0}{\sigma \omega^2}}$$

and

$$C = \frac{g}{\omega^2}$$

we get

$$x_{1,2} = -C \pm A$$

$$x_{3,4} = -C \pm B.$$

and since $B > A$, the roots are, in order of magnitude:

$$a_1 = x_4 = -C - B$$

$$a_2 = x_2 = -C - A$$

$$a_3 = x_1 = -C + A$$

$$a_4 = x_3 = -C + B.$$

Thus, in the order of magnitude given, $a_1 < a_2 < a_3 < a_4$.

For the purpose of transforming the elliptical integral to canonical form, be

$$m_1 = \sqrt{a_{13} \cdot a_{24}} \text{ and}$$

$$m_2 = \sqrt{a_{13} \cdot a_{34}}, \text{ where}$$

$$a_{12} = a_2 - a_1 = B - A$$

$$a_{13} = a_3 - a_1 = B + A$$

$$a_{24} = a_4 - a_2 = B + A$$

$$a_{34} = a_4 - a_3 = B - A.$$

With the use of the same

$$m_1 = \sqrt{(A + B)^2} = A + B \quad \text{and}$$

$$m_2 = \sqrt{(B - A)^2} = B - A.$$

If the values m_1 and m_2 are known, the modulus of the integral is given by the transformation:

$$k = \frac{m_1 - m_2}{m_1 + m_2} = \frac{2A}{2B} = \frac{A}{B} < 1,$$

as $B > A$.

If furthermore

$$m = \frac{m_1 + m_2}{2} = \frac{2B}{2} = B,$$

and

$$n_1 = \sqrt{a_{12} \cdot a_{24}}$$

$$n_2 = \sqrt{a_{13} \cdot a_{34}} \text{ and}$$

$$n = \frac{n_2 - n_1}{n_2 + n_1}, \quad \text{then}$$

$$n_1 = \sqrt{(B - A)(B + A)} = \sqrt{B^2 - A^2}$$

$$n_2 = \sqrt{(B + A)(B - A)} = \sqrt{B^2 - A^2}$$

$$n = \frac{0}{2\sqrt{B^2 - A^2}} = 0.$$

In order to transform the elliptical integral to canonical form, let us introduce the transformation:

$$x = \frac{1}{2}(a_3 + a_2) + \frac{1}{2}(a_3 - a_2) \cdot \frac{\sin\varphi - n}{1 - n \cdot \sin\varphi}.$$

We have, however,

$$C = -\frac{1}{2}(a_3 + a_2)$$

$$A = \frac{1}{2}(a_3 - a_2)$$

as

$$x = -C + A\sin\varphi.$$

For obtaining the canonical form of the elliptical integral, let us write in the denominator of the function

$$y = \int_0^x \frac{dx}{\sqrt{\left[\frac{\sigma}{V_0}\left(\frac{\omega^2}{2}x^2 + gx\right) - \frac{S_0}{V_0} + 1\right]\left[\frac{\sigma}{V_0}\left(\frac{\omega^2}{2}x^2 + gx\right) - \frac{S_0}{V_0} - 1\,\frac{-S^0}{V_0} - 1\right]}} + H$$

the polynomial under the root in form of a root factor. Eliminating the factors of the quadratic terms:

$$y = \frac{1}{\frac{\sigma\,\omega^2}{2V_0}} \int_0^x \frac{dx}{\sqrt{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}} + H$$

substituting the values of the radicals x_1, x_2, x_3 and x_4

$$y = \frac{2V_0}{\sigma\omega^2} \cdot \int_0^x \frac{dx}{\sqrt{(x+C-A)(x+C+A)(x+C-B)(x+C+B)}} + H.$$

Using the transformation $x = -C + A \sin\varphi$ for the integral, and since $dx = A \cdot \cos\varphi \, d\varphi$

$$y = \frac{2V_0}{\sigma\omega^2} \int_{\arcsin\frac{C}{A}}^{\arcsin\frac{x+C}{A}} \frac{A\cos\varphi\, d\varphi}{\sqrt{(A\sin\varphi - A)(A\sin\varphi + A)(A\sin\varphi - B)(A\sin\varphi + B)}} + H$$

or

$$y = \frac{2V_0}{\sigma\omega^2} \int_{\arcsin\frac{C}{A}}^{\arcsin\frac{x+C}{A}} \frac{A\cos\varphi\, d\varphi}{\sqrt{A^2B^2\left(\frac{A}{B}\sin\varphi - 1\right)\left(\frac{A}{B}\sin\varphi + 1\right)(\sin\varphi - 1)(\sin\varphi + 1)}} + H$$

i.e.

$$y = \frac{2V_0}{\sigma\omega^2} \int_{\arcsin\frac{C}{A}}^{\arcsin\frac{x+C}{A}} \frac{A\cos\varphi\, d\varphi}{A \cdot B\sqrt{\left(\frac{A^2}{B^2}\sin^2\varphi - 1\right)(\sin^2\varphi - 1)}} + H$$

after root extraction and simplification

$$y = \frac{2V_0}{\sigma\omega^2} \int_{\arcsin\frac{C}{A}}^{\arcsin\frac{x+C}{A}} \frac{d\varphi}{B \cdot \sqrt{\left(\frac{A^2}{B^2} \cdot \sin^2\varphi - 1\right) \cdot (-1)}} = H$$

substituting the value of the modulus $\frac{A}{B} = k$

$$y = \frac{2V_0}{\sigma\omega^2 B} \cdot \int_{\arcsin\frac{C}{A}}^{\arcsin\frac{x+C}{A}} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} + H.$$

By this transformation we obtain the canonical form of the integral, which is a Legendre-type elliptical integral of the first kind.

Thus, the equation of the balloon plane curve can be described by a Legendre-type elliptical integral of the first kind, which can be constructed with the help of Table F for elliptical integrals.

With the usual Legendre-type notation and in the case of a function of the following form

$$\mu = \int_0^{\eta} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = F(\eta, k)$$

η can be obtained from the expression

$$\eta = a_m(\mu, k).$$

Since furthermore

$$\int_{\alpha}^{\beta} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \int_{\alpha}^{0} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} + \int_{0}^{\beta} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} =$$

$$= \int_{0}^{\beta} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} - \int_{0}^{\alpha} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

hence the equation of the balloon plane curve

$$y = H + \frac{2V_0}{\sigma\,\omega^2 B} \cdot \int_{\arcsin \frac{C}{A}}^{\arcsin \frac{x+C}{A}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

can be written in the following form:

$$y = H + \frac{2V_0}{\sigma\omega^2 B} \cdot \left\{ \int_{0}^{\arcsin \frac{x+C}{A}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} - \int_{0}^{\arcsin \frac{C}{A}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \right\}.$$

Using again the notation

$$\sqrt{1 - k^2 \sin^2 \varphi} = \Delta\varphi$$

we get

$$y = H + \frac{2V_0}{\sigma\omega^2 B} \cdot \int_{0}^{\arcsin \frac{x+C}{A}} \frac{d\varphi}{\Delta\varphi} - \frac{2V_0}{\sigma\omega^2 B} \int_{0}^{\arcsin \frac{C}{A}} \frac{d\varphi}{\Delta\varphi}$$

or by deriving the integrals in the right side

$$\int_0^{\arcsin \frac{x+C}{A}} \frac{d\varphi}{\Delta\varphi} = F\left(\arcsin \frac{x+C}{A}, \quad k = \frac{A}{B}\right)$$

and

$$\int_0^{\arcsin \frac{C}{A}} \frac{d\varphi}{\Delta\varphi} = F\left(\arcsin \frac{C}{A}, \quad k = \frac{A}{B}\right)$$

the Legendre form of the equation of the balloon curve is obtained:

$$y = H + \frac{2V_0}{\sigma\omega^2 B} F\left(\arcsin \frac{x+C}{A}, \quad \frac{A}{B}\right) - \frac{2V_0}{\sigma\omega^2 B} F\left(\arcsin \frac{C}{A}, \quad \frac{A}{B}\right).$$

On the basis of the equation of the balloon curve formed by the unwinding yarn during weaving the tension arising in the weft yarn can be calculated. Determination of the tension will be discussed in a further paper.

Summary

Going out from the conditions of the dynamical equilibrium of the weaving balloon, the equation of the balloon plane curve formed by the unwinding yarn during weaving has been determined.

It is seen that even the equation of the balloon plane curve formed by the unwinding yarn in the shuttle during weaving cannot be derived by means of elementary functions. With an adequate transformation the canonical form of the curve can be obtained, a Legendre-type elliptical integral of the first kind.

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