periodica polytechnica

*Transportation Engineering*
*40/2 (2012) 67–70*
*doi: 10.3311/pp.tr.2012-2.04*
*web: http://www.pp.bme.hu/tr*


RESEARCH ARTICLE

# The limit of validity of the Newtonian fluid friction law in flows around vehicles and the limit of validity of the Navier-Stokes equations

*Egon* Jánosdeák
Received 2012-09-27

***Abstract***

*Known that the occurrence of shear stress is the result of the momentum exchange of particles (molecules in the case of gases), which takes place between the superimposed layers due to the thermal movement. This means that the forces corresponding to shear stress are arising actually in a very thin layer whose thickness is about the order of magnitude of the mean free path of the particles, so they are no surface forces in the strict sense, which is neglected in most cases.*

*We will show that there exists such a macroscopic flow at which this neglect is not allowed.*

***Keywords***

*flows around vehicles · non-stationary parallel planar · parabolic partial differential equation · Newtonian fluid friction law · Navier-Stokes equations*

**Egon Jánosdeák**

Department of Aircraft and Ships, BME, Stoczek u. 6. H-1111 Budapest, Hungary

In terms of the well known Newtonian fluid friction law shear stress arises between two superimposed moving fluid layers Fig. 1a which is given by the formula

$$\tau = \mu \frac{\mathrm{u}}{\mathrm{d}y} \tag{1}$$

However, it is also known that the occurrence of shear stress is the result of the momentum exchange of particles (molecules in the case of gases), which takes place between the superimposed layers due to the thermal movement [1–4] This means that the forces corresponding to shear stress are arising actually in a very thin layer whose thickness is about the order of magnitude of the mean free path of the particles, so they are no surface forces in the strict sense, which is neglected in most cases.

We will show that there exists such a macroscopic flow at which this neglect is not allowed.

Let us consider a non-stationary parallel planar flow according to Fig. 1a and Fig. 1b and write the equation of the dynamic equilibrium of the fluid element according to Fig. 2, temporarily excluding the consideration of the momentum exchange zone:

$$\rho \,\mathrm{d}x \,\mathrm{d}y \left(u_t + u_{yt}\frac{\mathrm{d}y}{2}\right) = \mu \left[\left(u_y + u_{yy}\,\mathrm{d}y\right) - u_y\right] \mathrm{d}x \tag{2}$$

where $\rho$ denotes the density, $\mu$ the dynamic viscosity, $u$ the velocity. After simplification and introduction of the kinematic viscosity

$$\nu = \frac{\mu}{\rho}$$

we obtain the differential equation

$$u_t + \frac{\mathrm{d}y}{2} u_{yt} = \nu u_{yy} \tag{3}$$

Given the fact that $\mathrm{d}y$ is differentially small, therefore it can be assumed that in the limit it equals zero, after rearranging we obtain finally the equation:

$$\nu u_{yy} - u_t = 0 \tag{4}$$

According to the criterion given in the appendix this is a parabolic partial differential equation, such equations describe however – as it is known – effects with infinite velocities of propagation (their characteristics slope is infinite). However, this

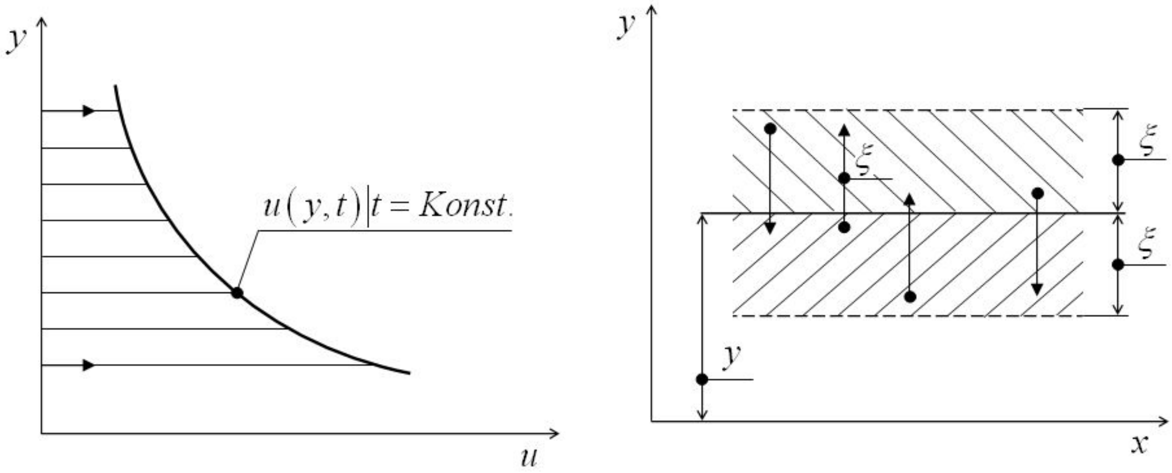


**Fig. 1.**

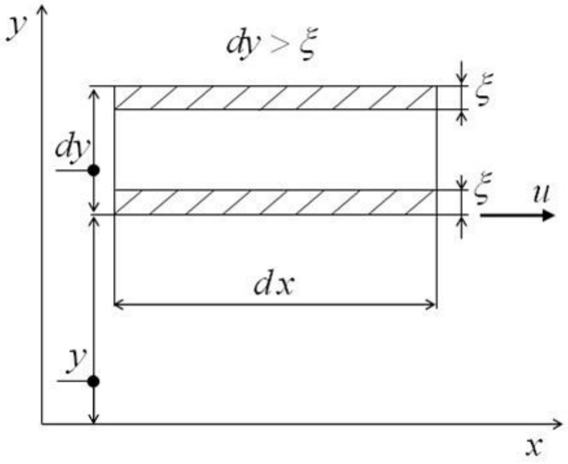


**Fig. 2.**

means that – regardless of relativistic considerations – Eq. (4) is untenable.

The situation is different when using Eq. (3) the boundary transition will not continue to $\mathrm{d}y = 0$ on the grounds that $\mathrm{d}y$ is not considered as a differential in the mathematical sense. Namely, we consider that the chosen $\mathrm{d}y$ thickness of the fluid element can't be less than the $\xi$ thickness of the standard surface layer from the viewpoint of the momentum exchange (Fig. 2), otherwise the forces corresponding the shear stress could not form. As it is shown in Appendix 2 this thickness can be regarded as approximately the mean free path of the molecules between two successive collisions:

$$\mathrm{d}y = \xi \tag{5}$$

thus we obtain from Eq. (3) the following equation:

$$\nu u_{yy} = u_t + \frac{\xi}{2} u_{yt} \tag{6}$$

If we assume that this is the laminar flow of a constant density gas, then with further adoption of the known relationship of the kinetic theory of gases:

$$\mu = 0,499 \rho c \xi \tag{7}$$ [1]

i.e.

$$\nu = \frac{\mu}{\rho} = 0,499 c \xi \tag{8}$$

where $c$ denotes the mean velocity of the thermal motion of the molecules we obtain the following hyperbolic equation:

$$0,499 c \xi u_{yy} - \frac{\xi}{2} u_{yt} = u_t \tag{9}$$

If we take with good approximation that $20,499 \cong 1,0$ we obtain:

$$c u_{yy} - u_{yt} = \frac{2}{\xi} u_t \tag{10}$$

It is well known that the hyperbolic equations describe phenomena characterized by finite propagation velocity. The general determination of the propagation velocities can be carried out with the well-known method of characteristics from the theory of hyperbolic equations. Since the purpose of the paper does not extend beyond the definition of the problem, we restrict ourselves to the illustrative specific solution of the Eqs. (4) and (10).

Eq. (4) with the initial condition $u(y, 0) = 0$ and constraint $u(0, t) = u_0 1(t)$ results in the solution (11)),

$$u = u_0 \operatorname{erfc} \frac{y}{2\sqrt{\nu t}} \tag{11}$$

which requires an infinite propagation velocity, because the $erfc$ function takes a finite positive value for each $0 < t < \infty$ and $0 < y < \infty$ values.

A possible solution to the Eq. (10) is

$$u = C' e^{\frac{1}{\xi}(ct-y)} \tag{12}$$

where $C'$ is a constant according to the initial and boundary conditions, which is uninteresting from our point of view. Function

[1] See: *Dr. Simonyi Károly: „Kinetikus gázelmélet, klasszikus statisztika" Egyetemi Nyomda, Budapest, 1949., p29, formula (15) etc.*

(12) describes a transverse "wave" (with reference to the formation of shear stresses, but no real wave in the true sense of the word) with $c$ velocity. Given the fact that at the temperature of *15° C* the average molecular thermal velocity for oxygen is 474m $s^{-1}$, for nitrogen 506m $s^{-1}$, the resulting propagation velocities for air appear to be in a plausible relationship with the speed of sound (longitudinal wave) in the air at the same temperature (340m $s^{-1}$).

After all the above, we write the $x$ directional component of the Navier-Stokes equation for the here considered constant density unsteady flat flow:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u,$$

where in our case $p$ is constant, the gravity component $\frac{\partial U}{\partial x} = 0$, while the $y$ and $z$ directed velocity components, respectively $v$ and $w$ are also equal zero.

After substitution, taking into account the usual interpretation of $\Delta$ and changing the notation of the derivatives according to $\frac{\partial u}{\partial y} = u_y$ and $\frac{\partial u}{\partial t} = u_t$ we get the following equation

$$\nu u_{yy} - u_t = 0 \tag{13}$$

which is identical to the untenable Eq. (4). Ultimately, Newton's law which is considered valid regardless of the geometric dimensions, in the case of application of the Navier-Stokes equations as the starting point resulted in the same untenable differential equation, such as the less general, physically more descriptive approach.

All in all, it seems that a revision of the the Navier-Stokes equations cannot be avoided, or there must be at least such criteria established which can secure that the application is allowable from the point of view of the sufficient accuracy of the results obtained in the specific cases characterized by the initial and boundary conditions.

The urgent topicality of the present paper is supported by an, in our view, unacceptable aim practically quoted from an English journal from the year 2003, *Mechanics & Thermodynamics, Journal of Mathematical Fluid Mechanics (Journal no. 21. Springer Verlag)*: *Description, Aims and Scope*

*The Journal of mathematical fluid Mechanics is a forum for the publication of high quality peer reviewed papers on the mathematical theory of fluid mechanics, with special regards to the Navier-Stokes equations. As an important part of that, the Journal encourages papers dealing with mathematical aspects of computational theory, as well as with applications in science and engineering.*

## Appendix

### 1. The criterion of the type of the here considered differential equations

Let us consider a differential equation of the type

$$\overline{A}u_{yy} + 2\overline{B}u_{yt} + \overline{C}u_{yy} + \overline{a}u_y + \overline{b}u_t + \overline{c}u = f(y,t)$$

where $\overline{A}, \overline{B}, \overline{C}, \overline{a}, \overline{b}, \overline{c}$ are coefficients independent from $y$ and $t$, $f(y,t)$ is a given function. If we form the discriminant

$$\delta = \overline{AC} - \overline{B}^2$$

the equation is elliptic, parabolic or hyperbolic according to

$$\delta = 0$$

In the case of Eq. (4) we get

$\overline{A} = \nu$; $\quad \overline{B} = \overline{C} = \overline{a} = \overline{c} = 0$; $\quad \overline{b} = -1$, that is

$\delta = \nu.0 - 0^2 = 0$, namely Eq. (4) is parabolic.

In the case of Eq. (10) $\overline{A} = \overline{c}; \overline{B} = -\frac{1}{2}; \overline{C} = \overline{a} = \overline{c} = 0; \overline{b} = -\frac{2}{\xi}; f(y,t) = 0$, wherewith

$\delta = c.0 - \left(-\frac{1}{2}\right)^2 = -\frac{1}{4}$, which means that Eq. (10) is hyperbolic.

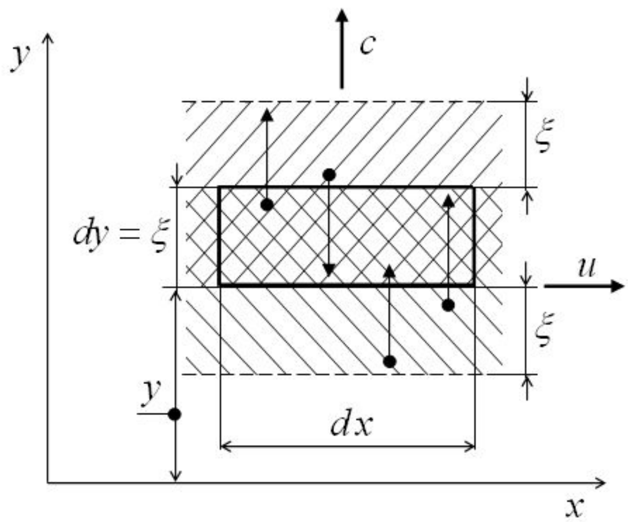


**Fig. 3.**

### 2. Reflections on the assumption $dy = \xi$

Let us consider a plane at a height of $y$ according to Fig. 1b. The shear stress acting on its each side, but in opposite direction according to Newton's law is

$$\tau = \mu\frac{\mathrm{u}}{\mathrm{d}y} \tag{14}$$

where the dynamic viscosity is given by the kinetic theory of gases by the formula

$$\mu = 0.499\rho c\xi \tag{15}$$

In this formula $\rho$ is the density, $c$ is the average thermal velocity of the molecules, $\xi$ is the mean free path of the molecules between two successive collisions.

In the majority of cases we neglect the fact that the force corresponding to the shear stress does not act in reality on the surface, but on a layer whose thickness is about $\xi$ and covers the surface on the side where the shear stress is considered to act (Fig. 1b).

If we now examine the dynamic equilibrium of a fluid element, in which the inner layer thicknesses belonging to boundary surfaces in height $y$ or $y + \mathrm{d}y$ cover each other exactly, then $\mathrm{d}y$ must equal $\xi$ and in the case of larger scales the impulses apparently generating shear stress act on the entire mass of the element. Upon further reduction of dimension $\mathrm{d}y$ the number of those molecules, which pass through the fluid element without collision and so without impulse emission, increases (Fig. 3).

### Designations

$x, y, z$ space coordinates
$u, v, w$ velocities corresponding space coordinates
$t$ time
$\rho$ density
$\nu$ kinematic viscosity
$\mu$ dynamic viscosity
$\xi$ mean free path of the gas molecules $c$ mean thermal velocity of the gas molecules
$C$ constant
$p$ pressure
$\overline{A}, \overline{B}, \overline{C}, \overline{a}, \overline{b}, \overline{c}$ constants
erfc = 1-erf
erf error function
$1\,(t)$ unit step function
$\overline{u}_0$ velocity step on the boundary surface ($x = 0; y = 0 \div \infty$) at $t = 0$

### References


1 **Kaulakytë K, Pileckas K**, *On the Nonhomogeneous Boundary Value Problem for the Navier–Stokes System in a Class of Unbounded Domains*, Journal of Mathematical Fluid Mechanics **14** (December 2012), 693–716, DOI 10.1007/s00021-011-0089-3.

2 **Feireisl E, Bum Ja Jin, Antonín Novotný**, *Relative Entropies, Suitable Weak Solutions, and Weak-Strong Uniqueness for the Compressible Navier–Stokes System*, Journal of Mathematical Fluid Mechanics **14** (December 2012), 717–730, DOI 10.1007/s00021-011-0091-9.

3 **Veress Á, Sánta I**, *A 2D Mathematical Model on Transonic Axial Compressor Rotor Flow*, Periodica Polytechnica Transportation Engineering **30** (2002), 53–68.

4 **Beneda K**, *Preliminary results of active centrifugal compressor surge control using variable inducer shroud bleed*, Periodica Polytechnica Transportation Engineering **39** (2011), 49–54, DOI 10.3311/pp.tr.2011-2.01.

5 **Budik Gy**, *Conversion of internal combustion engine from gasoline to E85 fuel*, Periodica Polytechnica Transportation Engineering **38** (2010), 19–23, DOI 10.3311/pp.tr.2010-1.04.